Introduction

The theory set out in this book is the result of the confluence and common development of two currents of mathematical research, Descriptive Set Theory and Recursion Theory. Both are concerned with notions of definability and, more broadly, with the classification of mathematical objects according to various measures of their complexity. These are the common themes which run through the topics discussed in this book.

Descriptive Set Theory arose around the turn of the century as a reaction among some of the mathematical analysts of the day against the free-wheeling methods of Cantorian set theory. People such as Baire, Borel, and Lebesgue felt uneasy with constructions which required the Axiom of Choice or the set of *all* countable ordinals and began to investigate what part of analysis could be carried out by more explicit and constructive means. Needless to say, there was vigorous disagreement over the meaning of these terms. Some of the landmarks in the early days of Descriptive Set Theory are the construction of the Borel sets, Suslin's Theorem that a set of real numbers is Borel just in case both it and its complement are analytic ($\mathbf{\Sigma}^1_1$), and the discovery that analytic sets have many pleasant properties — they are Lebesgue measurable, have the Baire property, and satisfy the Continuum Hypothesis.

A natural concomitant of this interest in the means necessary to effect mathematical constructions is the notion of hierarchy. Roughly speaking, a hierarchy is a classification of a collection of mathematical objects into levels, usually indexed by ordinal numbers. Objects appearing in levels indexed by larger ordinals are in some way more complex that those at lower levels and the index of the first level at which an object appears is thus a measure of the complexity of the object. Such a classification serves both to deepen our understanding of the objects classified and as a valuable technical tool for establishing their properties.

A familiar example, and one which was an important model in the development of the theory, is the hierarchy of Borel sets of real numbers. This class is most simply characterized as the smallest class of sets containing all intervals and closed under the operations of complementation and countable union. In the ρ-th level of the hierarchy are put sets which require a sequence of ρ

applications of complementation and countable union to families of intervals for their construction. The open and closed sets make up the first level, the F_σ (countable union of closed) and G_δ (countable intersection of open) sets the second, etc. This yields an analysis of the class of Borel sets into a strictly increasing sequence of $\aleph_1$ levels (see § V.3).

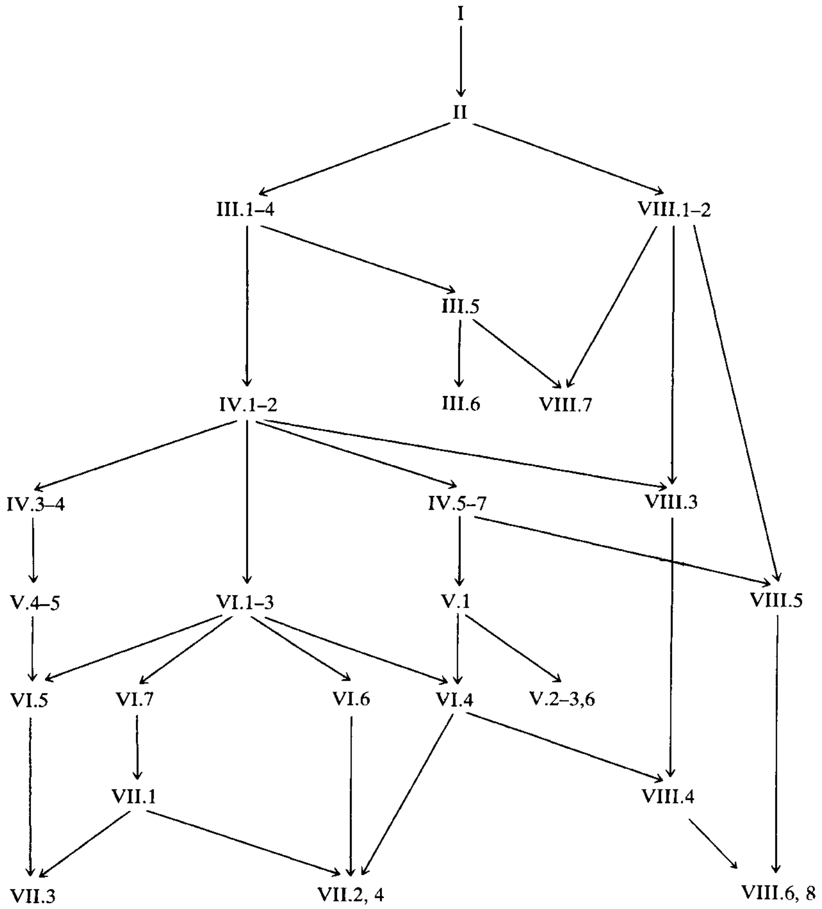
Recursion Theory developed in the 1930's as an attempt to give a rigorous meaning to the notion of a mechanically or algorithmically calculable function. Such a function is, in an obvious sense, more constructive and less complex than an arbitrary function. The first great success of Recursion Theory was Gödel's application of it in his incompleteness theorems in 1931. The diverse characterizations of the class of recursive functions by Church, Kleene, and Turing suggested strongly that this is a natural class of functions. Other related notions of (relative) complexity developed in the 1940's and 1950's — the notion of one function being recursive in another, the arithmetical and analytical hierarchies of Kleene and Mostowski, various sorts of definability in formal languages, and inductive definability.

The two theories developed essentially independently until the middle 1950's when, largely through the work of Addison, it was realized that to a great extent they are both special cases of a single general theory of definability. The results and methods of Recursion Theory are based on a more restrictive notion of constructivity and in many instances may in hindsight be viewed as refinements of their counterparts in Descriptive Set Theory. Since this is a book of mathematics rather than history, I shall develop here the general theory from which the results of both areas can be derived. As a result, many of the earlier parts of Descriptive Set Theory appear to depend on recursion-theoretic techniques. This dependence is explained by the fact that many of these techniques were known in some form before the advent of Recursion Theory. In a few cases (e.g. the Borel hierarchy) where the older theory is much simpler and more elegant than its recursion-theoretic refinement, I adopt a historical approach and present the classical version first. Roughly half of the material of the book comes from the period when the theories were separate; the other half is a product of the marriage.

In accord with the aims of this series, this book is a perspective on Recursion and Hierarchy Theory and not an encyclopedic treatment. Certain approaches are stressed heavily and other equally valid ones are omitted entirely. My definition of recursive function(al) in § II.2 is non-standard and is chosen for the ease with which it can be adapted to the definition of various classes of generalized recursive function(al)s in Chapters VI–VIII. Inductive definability is portrayed (correctly!) as the cornerstone of almost every aspect of the theory. Many topics closely related to those included are omitted. Some of the most notable of these are degree theory (in both ordinary and generalized recursion theory), abstract recursion theory (over other than the "natural" structures), axiomatic recursion theory, and subrecursive hierarchies. Most of these will be treated in other books in this series.

The book is intended for a variety of audiences. As a whole, it is aimed at a student with some general background in abstract mathematics — at least a smattering of topology, measure theory, and set theory — who has finished a course in logic covering the completeness and incompleteness theorems. In general, those parts which are more recursion-theoretic rely more heavily on a background in logic, while the more descriptive set-theoretic parts use more topology. Of course, as always, lack of formal experience in any area is compensated for by that elusive "mathematical maturity". The book may be used for a variety of formal courses of study under titles such as (Generalized) Recursion Theory, Descriptive Set Theory, or Theory of Definability. Students with sufficient background to skim Chapters I and II quickly can cover most of the book in a full-year course, but otherwise some judicious pruning will be required.

In general, the sections of the book depend on each other as indicated in the following diagram; some individual results may presuppose more or less background.



In designing the proofs of results in this book, I have in general made the assumption that the reader has access to pen and paper and will not mind working out a few points for him/herself, but in the main I have tried to give sufficient detail so that anyone who has mastered the prerequisite sections will find this an easy exercise. The subject is full of proofs which require a rather intricate construction followed by a tedious but straightforward inductive verification that the object constructed does the job it was designed for. Furthermore, one often has need in later parts of the theory for a construction quite similar to an earlier one but with one or two additional twists. Indeed, I have often arranged a sequence of lemmas and theorems exactly so that the constructions become increasingly complex in stages in preference to giving at once the most general case. In presenting these I have tried to strike a middle ground between putting the reader to sleep by constant repetition and overly free use of that attractive term "obviously". In general, the construction, or the modification of an earlier construction, is given in full detail, but a good part of the verification is left to the reader. I suspect that the average reader will usually be content to know that the rest of the proof runs "similarly", while the devoted reader (if any!) who seeks a firm grasp on the methods of the subject as well as the results will benefit from the process of working out the details.

The exercises are of two main types. The more routine among them are designed to give the student some experience in handling the methods and techniques of the text and usually require few new ideas. These occasionally include a proof of a lemma from the section. Many of the exercises, however, present results which might well have been included in a larger or more specialized book and constitute a do-it-yourself supplement to the book. (Indeed, one mathematician was somewhat offended that his favorite theorem achieved only the status of an exercise!) I have provided hints and suggestions where they seemed necessary, but many of these exercises will be quite challenging even to the experienced student. The more casual reader should at least glance over the exercises for statements of results.

With respect to the history of the subject, I have taken a middle course between suppressing it altogether and trying to document and credit each minute advance. The primary purpose of the Notes at the end of most sections is to give some idea how, when, and by whom the subject (was) developed, but in the interest of brevity I have omitted mention of many significant contributors. I apologize to those slighted and hope that they will recognize that their sacrifice is for a good cause.

The References similarly contain only a fraction of the articles and books in which our subject matured. Many older references are in a style so different from the current one that they are of little practical use to the working mathematician. I have included a few of these for their historical importance, but in the main the works cited are ones I feel might be of interest to the serious student. In some cases they contain material beyond that of the text, in others they will provide further insight into origins and motivations. In the Epilogue I discuss the current literature and give some guidance for reading which goes beyond the confines of this book.