The Core Model Iterability Problem

J. R. Steel

These notes develop a method for constructing *core models*, that is, canonical inner models of the form $L[\boldsymbol{E}]$, where $\boldsymbol{E}$ is a coherent sequence of extenders. They extend the earlier work in this area of Dodd and Jensen ([DJ1], [DJ2], [DJ3]) and Mitchell ([M1], [M?]). The Dodd-Jensen theory produces models having measurable cardinals, and Mitchell's extension of it produces models having measurable cardinals κ of order κ^{++}. Here we shall extend this theory so that it can produce core models having Woodin cardinals.

The extent of our debt to Dodd, Jensen, and Mitchell will become apparent; nevertheless, we shall not assume that the reader is familiar with their work. We shall, however, assume that he is familiar with the fine structure theory for core models having Woodin cardinals which is developed in [FSIT].

Our work here goes beyond [FSIT] in that it involves a construction of $L[\boldsymbol{E}]$ models which makes no use of extenders over V. Many of the applications of core model theory require such a construction. The authors of [FSIT] use extenders over V in order to show that the inner model $L[\boldsymbol{E}]$ they construct is sufficiently iterable: roughly, they demand that the extenders put onto $\boldsymbol{E}$ be the restrictions of background extenders over V, then use this fact to embed iteration trees on $L[\boldsymbol{E}]$ into iteration trees on V, and then quote the results of [IT] concerning iteration trees on V. Here we shall describe a weakened background condition on the extenders put onto $\boldsymbol{E}$ which does not require full extenders over V, and yet suffices to carry out something like the old proof of the iterability of $L[\boldsymbol{E}]$. The result is a solution to what is called the "core model iterability problem" in [FSIT].

The notes are organized as follows. As in Mitchell's work on the core model for sequences of measures ([M1], [M?]), we construct the model K in which we are ultimately most interested in two steps. In §1 we construct a model, which we call K^c, whose extenders have "background certificates". These background certificates guarantee the iterability of K^c, its levels, and various associated structures. (In Mitchell's work, the background condition is countable completeness, but here we seem to need more.) In order to show that K^c and the K we derive from it are large enough to be useful, we seem to need something like the existence of a measurable cardinal. We fix a normal measure μ_0 on a measurable cardinal Ω throughout this paper; we shall have $\mathrm{OR} \cap K^c = \mathrm{OR} \cap K = \Omega$. We use the measurability of Ω to show in §1 that either $K^c \models$ there is a Woodin cardinal, or $(\alpha^+)^{K^c} = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$. Since in applications we are seeking an inner model with a Woodin cardinal, we assume through most of the paper that there is no such model, and thus we have $(\alpha^+)^{K^c} = \alpha^+$ for μ_0- a.e. $\alpha < \Omega$. As in Mitchell's work, this "weak covering property" of K^c is crucial.

We also use the measurability of Ω in a different way in §4. Further, we use the tree property of Ω to show that iteration trees of length Ω are well behaved. The fact that we do not develop the basic theory of K within ZFC may be a defect in our work.

In §2 we sketch the main new ideas in the proof that K^c is iterable. In §9 we give a full proof of a general iterability theorem which covers iteration trees and psuedo-iteration trees on K^c, its levels, and the associated bicephali and psuedo-premice.

As in Mitchell's work, the "true core model" K is a Skolem hull of K^c. In §3 and §4 we develop some concepts, derived from Mitchell's work, which are useful in the construction of this hull. In §5 we do the construction: given a stationary $S \subseteq \Omega$ with certain properties, we construct a model $K(S) \preceq K^c$. We show $(\alpha^+)^{K(S)} = \alpha^+$ for μ_0- a.e. α. We also show that $K(S)$ is invariant under small forcing; that is, $K(S) = K(S)^{V[G]}$ whenever G is V generic over some $\mathbb{P} \in V_\Omega$. Finally, we show that $K(S)$ is independent of S, and define K to be the common value of $K(S)$ for all S. We have then that $(\alpha^+)^K = \alpha^+$ for μ_0 - a.e. α, and that K is invariant under set forcing.

In §6 we give an optimally simple inductive definition of K: it turns out that $K \cap HC$ is $\Sigma_1(L_{\omega_1}(\mathbb{R}))$. (Woodin has shown that no simpler definition is possible in general. Mitchell showed in [M?] that if no initial segment of K satisfies $(\exists\kappa)(o(\kappa) = \kappa^{++})$, then $K \cap HC$ is Σ^1_5 in the codes.) In §7 we use the machinery we have developed to obtain the consistency strength lower bound of one Woodin cardinal for various propositions. In §8 we return to the pure theory, and obtain some information concerning embeddings of K. We show, for example, that if there is no inner model with a Woodin cardinal, then there is no nontrivial elementary $j : K \to K$. In contrast to the situation for "smaller K's", however, we show that there may be nontrivial elementary $j : K \to M$ which are not iteration maps.

Among the applications of the theory developed in these notes are the following theorems.

Theorem 0.1. *Let Ω be measurable, and suppose there is a presaturated ideal on ω_1; then there is a transitive set $M \subseteq V_\Omega$ such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

Corollary 0.2. *If Martin's Maximum holds, then there is a transitive set M such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

(It is known from [FMS] that Martin's Maximum implies that there is an inner model with a measurable cardinal and a saturated ideal on ω_1; by applying 0.1 inside this model we get 0.2. H. Woodin pointed this out to the author.)

Further work of Mitchell, Schimmerling, and the author on the weak covering property for K (cf. [WCP]) together with his work on Jensen's $\Box$ principle in K (cf. [Sch]), led Schimmerling to the following improvement of 0.2.

Theorem 0.3. (Schimmerling, cf. [Sch]) *If PFA holds, then there is a transitive set M such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

(Theorem 0.3 also relies on an improvement, due to Magidor, of Todorčević's result that PFA implies $\forall\kappa$ ($\Box_\kappa$ fails). (See [To].)

In a different vein, we have the following, more immediate applications of the theory presented here.

Theorem 0.4. *Suppose that every set of reals which is definable over $L_{\omega_1}(\mathbb{R})$ is weakly homogeneous; then there is a transitive set M such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

(H. Woodin supplied a crucial step in the proof of 0.4.) Since Woodin (unpublished) has shown that the existence of a strongly compact cardinal implies the hypothesis of 0.4, we have

Corollary 0.5. *Suppose there is a strongly compact cardinal; then there is a transitive set M such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

(We shall give a different proof of 0.5, one which avoids Woodin's unpublished work, in §8.)

We can also improve the lower bound of [IT] on the strength of the failure of UBH.

Theorem 0.6. *Let Ω be measurable, and suppose there is an iteration tree $\mathcal{T}$ on V_Ω such that $\mathcal{T} \in V_\Omega$ and $\mathcal{T}$ has distinct cofinal wellfounded branches; then there is a transitive set $M \subseteq V_\Omega$ such that*

$$M \models ZFC + \text{"There are two Woodin cardinals"}.$$

We can use the methods presented here to re-prove Woodin's result that $\forall x \in^\omega \omega(x^\sharp$ exists) + Δ^1_2 determinacy implies that there is an inner model with a Woodin cardinal. Finally, using an idea of G. Hjorth, the author has recently (at least partially) generalized Jensen's Σ^1_3 correctness theorem to the core model constructed here. This yields a positive answer to a conjecture of A. S. Kechris.

Theorem 0.7. *Assume $\forall x \in {}^\omega\omega$ ($x^\sharp$ exists and $\Sigma^1_3(x)$ has the separation property); then there is a transitive set M such that*

$$M \models ZFC + \text{"There is a Woodin cardinal"}.$$

Each of the hypotheses of the theorems above is known to be consistent under some large cardinal hypothesis or other. We shall not attempt a scholarly discussion of the history of or context for these theorems, as our focus here is the basic theory which produces them. We shall prove 0.1, 0.4, 0.6, and 0.7 in §7.

Historical note. We did most of the work described here in the Spring of 1990, and informally circulated it in a set of handwritten notes. Our main advance, which was isolating K^c and proving the results of §1 and §2 concerning it, was inspired in part by some ideas of Mitchell. The work in §8 was done somewhat later, and the Σ^1_3 correctness theorem of §7D was not proved until the Spring of 1993.

§1. The construction of K^c

Let us fix a measurable cardinal, which we call Ω, for the remainder of this paper. We shall sometimes think of Ω as the class of all ordinals; we could have worked in 3rd order set theory + "OR is measurable", but opted for a little more room. Fix also a normal measure μ_0 on Ω.

We now define by induction on $\xi < \Omega$ premice $\mathcal{N}_\xi$. Having defined $\mathcal{N}_\xi$, we set

$$\mathcal{M}_\xi = \mathfrak{C}_\omega(\mathcal{N}_\xi)\,,$$

the ωth core of $\mathcal{N}_\xi$. The $\mathcal{M}_\xi$'s will converge to the levels of K^c, the "background-certified" core model for one Woodin cardinal. That is, we shall define K^c by setting

$$\mathcal{J}_\beta^{K^c} = \text{ eventual value of } \mathcal{J}_\beta^{\mathcal{M}_\xi}\,, \quad \text{as } \xi \to \Omega\,,$$

for all $\beta < \Omega$. The construction of the $\mathcal{N}_\xi$'s follows closely the construction in §11 of [FSIT].

In this section, we shall simply assume that the $\mathcal{N}_\xi$'s are all "reliable", that is, that $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists and is k-iterable for all $k \leq \omega$. By Theorem 8.1 of [FSIT], this amounts to assuming that if $\mathfrak{C}_k(\mathcal{N}_\xi)$ exists, then it is k-iterable (for all $k \leq \omega$). We shall sketch a proof of this iterability assumption in §2, and give a full proof in §9. We shall also assume here that certain bicephali and psuedo-premice associated to $\langle \mathcal{N}_\xi \mid \xi < \Omega \rangle$ are sufficiently iterable. We prove this in §9.

Iterability comes from the existence of background extenders.

Definition 1.1. *Let $\mathcal{M}$ be an active premouse, F the extender coded by $\dot{F}^{\mathcal{M}}$ (i.e. its last extender), $\kappa = crit(F)$, and $\nu = \dot{\nu}^{\mathcal{M}} = \sup$ of the generators of F. Let $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$. Then an $\mathcal{A}$-certificate for $\mathcal{M}$ is a pair (N, G) such that*

(a) *N is a transitive, power admissible set, $V_\kappa \cup \mathcal{A} \subseteq N$, ${}^\omega N \subseteq N$, and G is an extender over N,*

(b) *$F \cap ([\nu]^{<\omega} \times \mathcal{A}) = G \cap ([\nu]^{<\omega} \times \mathcal{A})$,*

(c) *$V_{\nu+1} \subseteq Ult(N, G)$, and ${}^\omega Ult(N, G) \subseteq Ult(N, G)$,*

(d) *$\forall \gamma(\omega\gamma < OR^{\mathcal{M}} \Rightarrow \mathcal{J}_\gamma^{\mathcal{M}} = \mathcal{J}_\gamma^{i(\mathcal{J}_\kappa^{\mathcal{M}})})$, where $i : N \to Ult(N, G)$ is the canonical embedding.*

Definition 1.2. *Let $\mathcal{M}$ be an active premouse, and κ the critical point of its last extender. We say $\mathcal{M}$ is countably certified iff for every countable $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$, there is an $\mathcal{A}$-certificate for $\mathcal{M}$.*

In the situation described in 1.2 we shall typically have $|N| = \kappa$, so that $(\text{OR} \cap N) < lhG$. We are therefore not thinking of (N, G) as a structure to be iterated; N simply provides a reasonably large collection of sets to be measured by G. The conditions $V_\kappa \subseteq N$ and $V_{\nu+1} \subseteq \text{Ult}(N, G)$ are crucial; the closure of N and $\text{Ult}(N, G)$ under ω-sequences could probably be dropped.

It might seem that certificates (N, G) as in 1.1 are too much to ask for, and in particular condition 1.1 (c) might seem too strong. But notice that we get such pairs by taking Skolem hulls: if $\pi : N \cong H \prec V_\eta$ inverts the collapse of H, where $V_\kappa \subseteq H$ but $\kappa \notin H$, then letting G be the length $\pi(\kappa)$ extender derived from π, G is an extender over N, $V_\kappa \subseteq N$, and $V_{\pi(\kappa)} \subseteq \mathrm{Ult}(N, G)$. We shall also see that the embedding associated to the measure μ_0 on Ω gives rise to certificates.

Definitions 1.1 and 1.2 were inspired in part by earlier attempts by Mitchell to formulate a background condition along these lines.

We proceed to the inductive definition of $\mathcal{N}_\xi$. As we define the $\mathcal{N}_\xi$'s we verify:

A_ξ : $\forall \alpha < \xi\ \forall \kappa$ (if $\rho_\omega(\mathcal{M}_\gamma) \geq \kappa$ for all γ s.t. $\alpha < \gamma \leq \xi$, then letting $\eta = (\kappa^+)^{\mathcal{M}_\alpha}$, $\mathcal{J}_\eta^{\mathcal{M}_\alpha} = \mathcal{J}_\eta^{\mathcal{M}_\xi}$).

(Here we let $\omega\eta = \mathrm{OR} \cap \mathcal{M}_\alpha$ in the case $\mathcal{M}_\alpha \models \kappa^+$ doesn't exist.)

We begin by setting $\mathcal{N}_0 = (V_\omega, \in)$. (So $\mathcal{M}_0 = \mathcal{N}_0$.) Now suppose $\mathcal{N}_\xi$, and hence $\mathcal{M}_\xi$, is given.

Case 1. $\mathcal{M}_\xi = (J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma)$ is a passive premouse, and there is an extender F over $\mathcal{M}_\xi$ such that

(1) $(J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma, \tilde{F})$ is a 1-small, countably certified premouse, and, letting $\kappa = \mathrm{crit}(F)$, we have

(2) κ is inaccessible,

(3) $(\kappa^+)^{\mathcal{M}_\xi} = \kappa^+ \Rightarrow$ for stationary many $\beta < \kappa$, β is inaccessible and $(\beta^+)^{\mathcal{M}_\xi} = \beta^+$.

In this case, we choose an F as above with $\nu(F)$, the sup of the generators of F, as small as possible, and we set

$$\mathcal{N}_{\xi+1} = (J_\gamma^{\boldsymbol{E}}, \in, \boldsymbol{E} \restriction \gamma, \tilde{F}).$$

As we mentioned above, the results of §9 and §8 of [FSIT] imply that $\mathcal{N}_{\xi+1}$ is reliable. Thus $\mathcal{M}_{\xi+1} = \mathfrak{C}_\omega(\mathcal{N}_{\xi+1})$ exists. We get $A_{\xi+1}$ from (the proof of) Theorem 8.1 of [FSIT].

Case 2. Otherwise.

In this case, let $\omega\gamma = \mathrm{OR} \cap \mathcal{M}_\xi$, and set

$$\mathcal{N}_{\xi+1} = (J_{\gamma+1}^{\dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}}, \in, \dot{E}^{\mathcal{M}_\xi} \frown \dot{F}^{\mathcal{M}_\xi}).$$

Again, $\mathcal{N}_{\xi+1}$ is reliable by §9, and 8.1 of [FSIT] yields $A_{\xi+1}$.

So in Case 1 we add a countably certified extender to our extender sequence, while in Case 2 we take one step in the constructible closure of the sequence we have. In both cases we then form the ωth core of the structure we have.

Now suppose we have defined $\mathcal{N}_\xi$ for $\xi < \lambda$, where $\lambda < \Omega$ is a limit ordinal. Set

$$\eta = \liminf_{\xi \to \lambda} (\rho_\omega(\mathcal{M}_\xi)^{+\mathcal{M}_\xi})$$

(where $\rho_\omega(\mathcal{M}_\xi)^{+\mathcal{M}_\xi} = \mathrm{OR} \cap \mathcal{M}_\xi$ is possible. We set $\mathcal{N}_\lambda$ = unique passive premouse $\mathcal{P}$ s.t.

(a) $\forall \beta < \eta(\mathcal{J}_\beta^{\mathcal{P}}$ = eventual value of $\mathcal{J}_\beta^{\mathcal{M}_\xi}$ as $\xi \to \lambda)$ and (b) $\mathcal{J}_\eta^{\mathcal{P}} = \mathcal{P}$. Such a premouse exists as A_ξ holds for all $\xi < \lambda$. It is easy to verify A_λ.

This completes the inductive definition of the $\mathcal{N}_\xi$, for $\xi < \Omega$.

Theorem 8.1 of [FSIT] actually gives the following strengthening of our induction hypothesis A_η. (Cf. 11.2 of [FSIT].)

Lemma 1.3. *Suppose $\kappa \leq \rho_\omega(\mathcal{M}_\xi)$ for all $\xi \geq \alpha_0$, and let $\xi \geq \alpha_0$ be such that $\kappa = \rho_\omega(\mathcal{M}_\xi)$. Then $\mathcal{M}_\xi$ is an initial segment of $\mathcal{M}_\eta$, for all $\eta \geq \xi$; moreover $\mathcal{M}_{\xi+1}$ satisfies "every set has cardinality $\leq \kappa$".*

Lemma 1.3 implies $\liminf_{\xi \to \Omega} \rho_\omega(\mathcal{M}_\xi) = \Omega$, and therefore we can define a premouse K^c of ordinal height Ω by

$$\mathcal{J}_\beta^{K^c} = \text{eventual value of} \quad \mathcal{J}_\beta^{\mathcal{M}_\xi}, \quad \text{as} \quad \xi \to \Omega\,,$$

for all $\beta < \Omega$.

The following is a cheapo form of weak covering. It is crucial in what follows; it tells us that we haven't been too miserly about putting extenders on the K^c sequence.

Theorem 1.4. *Exactly one of the following holds*:

(a) $K^c \models$ *There is a Woodin cardinal,*

(b) *for* μ_0- *a.e.* $\alpha < \Omega$, $(\alpha^+)^{K^c} = \alpha^+$.

Proof. (a) $\Rightarrow$ $\neg$ (b): Every $\mathcal{J}_\beta^{K^c}$ is 1-small, so if $K^c \models \delta$ is Woodin, then $E_\gamma^{K^c} = \emptyset$ if $\gamma \geq \delta$. As Ω is measurable, there is a club class of indiscernibles for K^c, or equivalently, a countable mouse $\mathcal{N}$ which is not 1-small. Comparing $\mathcal{N}$ with K^c, we get that for μ_0- a.e. $\alpha < \Omega$, $(\alpha^+)^{K^c}$ has cofinality ω in V.

Remark. We have no use for this direction in what follows.

$\neg$ (b) $\Rightarrow$ (a):

Let $j : V \to M = \mathrm{Ult}(V, \mu_0)$ be the canonical embedding. We are assuming then

$$(\Omega^+)^{j(K^c)} < \Omega^+\,.$$

(Of course, $\Omega^{+M} = \Omega^+$.) Let $\mathcal{A} = P(\Omega) \cap j(K^c)$, so that $\mathcal{A} \in M$ and $M \models |\mathcal{A}| = \Omega$. Let E_j be the $(\Omega, j(\Omega))$ extender derived from j. By an ancient argument due to Kunen, whenever $|\mathcal{B}| = \Omega$, $E_j \cap ([j(\Omega)]^{<\omega} \times \mathcal{B}) \in M$.

(Proof: if $\mathcal{B} = \{B_\alpha \mid \alpha < \Omega\}$, then notice $\langle j(B_\alpha) \mid \alpha < \Omega\rangle = j(\langle B_\alpha \mid \alpha < \Omega\rangle) \restriction \Omega$, so $\langle j(B_\alpha) \mid \alpha < \Omega\rangle \in M$. Also, $B_\alpha \in (E_j)_c$ iff $c \in j(B_\alpha)$.)

In particular, setting

$$F = E_j \cap ([j(\Omega)]^{<\omega} \times j(K^c))\,,$$

we have that $F \in M$. Now it cannot be that every proper initial segment $F \restriction \nu$, $\nu < j(\Omega)$, of F belongs to $j(K^c)$, as otherwise these initial segments witness that Ω is Shelah in $j(K^c)$. But if $K^c \models$ There are no Woodins, then as $\mathcal{J}_{\Omega}^{j(K^c)} = K^c$, 1-smallness is not a barrier to adding these $F \restriction \nu$ to the $j(K^c)$ sequence. The following claim asserts that there are no other barriers. Its statement and proof run parallel to those of Lemma 11.4 of [FSIT].

Claim. Suppose $K^c \models$ There are no Woodin cardinals. Let $(\Omega^+)^{j(K^c)} \leq \rho < j(\Omega)$ and suppose that either $\rho = (\Omega^+)^{j(K^c)}$, or $\rho - 1$ exists and is a generator of F, or ρ is a limit of generators of F. Let G be the trivial completion of $F \restriction \rho$, and $\gamma = lhG$. Let $\boldsymbol{E}$ be the extender sequence of $j(K^c)$. Then either

(a) $\rho = (\Omega^+)^{j(K^c)}$, $\gamma \in \text{dom}\, \boldsymbol{E}$, and $E_\gamma = G = F \restriction \gamma$, or

(b) $\rho - 1$ exists, $\gamma \in \text{dom}\, \boldsymbol{E}$, and $E_\gamma = G = F \restriction \gamma$, or

(c) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$, ρ is not a generator of F, and $\gamma \in \text{dom}\, \boldsymbol{E}$ and $E_\gamma = G = F \restriction \gamma$, or

(d) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$ ρ is itself a generator of F, $\rho \notin \text{dom}\, \boldsymbol{E}$, and $\gamma \in \text{dom}\, \boldsymbol{E}$ and $E_\gamma = G$ (but $E_\gamma \neq F \restriction \gamma$), or

(e) ρ is a limit ordinal $> (\Omega^+)^{j(K^c)}$, ρ is itself a generator of F, $\rho \notin \text{dom}\, \boldsymbol{E}$, and $\pi(\boldsymbol{E})_\gamma = G$, where π is the canonical embedding from $J_\rho^{\boldsymbol{E}}$ to $\text{Ult}_0(J_\rho^{\boldsymbol{E}}, E_\rho)$.

Proof. By induction on ρ. As the proof is rather convoluted and follows closely the proof of Lemma 11.4 of [FSIT], we shall not give it here. The idea is as follows: we show that G satisfies, in M, all the conditions for being added to the $j(K^c)$ sequence as E_γ. We have 1-smallness by hypothesis, and coherence because F is a restriction of E_j. (If ρ is a generator of F, then we have to appeal to the condensation Theorem 8.2 of [FSIT] here.) We have the initial segment condition by induction. For our background certificates we can take (N, H), where $V_\Omega \cup \mathcal{A} \subsetneq N$, $|N| = \Omega$, and $H = E_j \cap ([j(\Omega)]^{<\omega} \times N)$. (As we remarked earlier, $H \in M$.) Since G can be added to $j(K^c)$ in M as E_j, the construction of $j(K^c)$ guarantees $\gamma \in \text{dom}\, \boldsymbol{E}$. A "bicephalus" or "Doddage" argument guarantees $G = E_\gamma$.

The foregoing is more or less a proof in the case (a) or (c) of the conclusion holds. If (b) holds, we run into technical problems with mixed type bicephali. If (d) or (e) holds, we also run into the problem that $\mathcal{J}_\gamma^{j(K^c)}$ may not be a stage $\mathcal{M}_\xi$ of the construction of $j(K^c)$ done within M. We refer the reader to §11 of [FSIT] for a full proof.

We should note that this argument requires the iterability of the bicephali and psuedo-premice which arise. (See §11 of [FSIT] for more detail on how they arise.) We shall prove this in §9. □

We define $A_0 \subsetneq \Omega$ by

$$\alpha \in A_0 \Leftrightarrow \alpha \text{ is inaccessible and } (\alpha^+)^{K^c} = \alpha^+ \text{ and}$$
$$\{\beta < \alpha \mid \beta \text{ is inaccessible and } (\beta^+)^{K^c} = \beta^+\} \text{ is not stationary in } \alpha\,.$$

One can easily check that if $K^c \models$ there are no Woodin cardinals, then

(i) A_0 is stationary in Ω,

(ii) A_0 has μ_0-measure 0,
(iii) $\alpha \in A_0 \Rightarrow ((\alpha^+)^{K^c} = \alpha^+ \wedge \alpha$ is inaccessible),
(iv) $\alpha \in A_0 \Rightarrow \alpha$ is not the critical point of any total-on-K^c extender from the K^c sequence.

It was in order to insure the existence of a set with the properties of A_0 that we included condition (3) in Case 1 of the construction of K^c. (Condition (2) could be dropped, but it does no harm.)

The definition of K^c which we have given in this section is unnatural in one respect: its requirement that the $\mathcal{N}_\xi$'s, and hence K^c itself, be 1-small. We believe that, were this restriction simply dropped, the resulting $\mathcal{N}'_\xi$'s would converge to a model $(K^c)'$ of height Ω, and one could show that either $(\alpha^+)^{(K^c)'} = \alpha^+$ for μ_0 a.e. $\alpha < \Omega$, or $(K^c)' \models$ there is a superstrong cardinal. What is missing at the moment is a proof that if M is a countable elementary submodel of one of the $\mathcal{N}'_\xi$'s or their associated psuedo-premice and bicephali, then M is $\omega_1 + 1$ iterable (in the sense of definition 2.8 of this paper). At the moment we can only prove this iterability result for mice which are "tame" (do not have extenders overlapping Woodin cardinals), and thus it is only to such mice that we can extend the theory presented here. (See [CMWC] and [TM].)

§2. Iterability

In this section we shall sketch a proof that if $\mathcal{T}$ is a k-maximal iteration tree on $\mathfrak{C}_k(\mathcal{N}_\xi)$, and $\mathcal{T} \restriction \lambda$ is simple for all limit $\lambda < lh\mathcal{T}$, and $lh\mathcal{T}$ is a limit ordinal, then for all sufficiently large κ, $V^{\mathrm{Col}(\omega,\kappa)} \models \mathcal{T}$ has a cofinal, wellfounded branch. There are several other iterability facts we shall need, and actually we shall not prove even this one in this section, since we shall make some simplifying assumptions on $\mathcal{T}$. The reader seeking full detail and generality will find it in §9. The reader who would like to see the main ideas in our iterability proof, while avoiding full detail and generality, can content himself with this section.

In this paper, we shall diverge slightly from the terminology of [FSIT] regarding iteration trees. By an *iteration tree* we mean a system $\mathcal{T}$ obeying all the conditions required in the definition of §5 of [FSIT] except possibly the increasing length condition. That is, we do not require $\alpha < \beta \Rightarrow lh\ E^{\mathcal{T}}_\alpha < lh\ E^{\mathcal{T}}_\beta$. Iteration trees in the sense of [FSIT] we call *normal.* (We note that even in a normal tree $\mathcal{T}$, $E^{\mathcal{T}}_\alpha$ may not be applied to the earliest possible model. This last requirement is part of k-maximality.) Although the trees which arise in comparison processes are all normal and k-maximal for some $k \leq \omega$, we must cover more than such trees in our proof that K^c is iterable. This is because the proof of the Dodd-Jensen lemma (5.3 of [FSIT]) involves non-normal trees.

We shall say that $\mathcal{T}$ is simple iff for all sufficiently large κ, $V^{\mathrm{Col}(\omega,\kappa)} \models \mathcal{T}$ has at most one cofinal wellfounded branch. (This diverges slightly from the terminology of [FSIT].) We shall need a relative of this notion.

Definition 2.1. *Let $\mathcal{T}$ be a tree on $\mathcal{M}$ of limit length, and $\alpha \in OR$. We say $\mathcal{T}$ is α-short iff for all sufficiently large κ*

$$V^{Col(\omega,\kappa)} \models \mathcal{T} \text{ has no cofinal branch } b \text{ such that } \alpha \text{ is isomorphic to an initial segment of } OR^{\mathcal{M}^{\mathcal{T}}_b}.$$

(Here $\mathcal{M}^{\mathcal{T}}_b = \lim_{\alpha \in b} \mathcal{M}^{\mathcal{T}}_\alpha$.)

The next two lemmas come from the uniqueness theorem of §2 of [IT]. See also Theorem 6.1 of [FSIT]. Their formulation also owes a lot to work of Woodin, and to the Π^1_2 mouse condition of §5 of [IT].

Lemma 2.2. *Let $\mathcal{M}$ be 1-small and $\mathcal{T}$ an iteration tree on $\mathcal{M}$, and let $\lambda < lh\ \mathcal{T}$. Then for some $\alpha \in OR$, $\mathcal{T} \restriction \lambda$ is α-short.*

Proof. Assume not. Let

$$\begin{aligned} \delta &= \sup\{lh\ E^{\mathcal{T}}_\beta \mid \beta < \lambda\}, \\ \vec{E} &= \bigcup_{\beta<\lambda} \dot{E}^{\mathcal{M}^{\mathcal{T}}_\beta} \restriction lh\ E^{\mathcal{T}}_\beta, \end{aligned}$$