Cambridge University Press
978-1-107-16748-3 — Algebraic Groups
The Theory of Group Schemes of Finite Type over a Field
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Excerpt
More Information

Introduction

The book can be divided roughly into five parts.

A. Basic theory of general algebraic groups (Chapters 1–8)

The first eight chapters cover the general theory of algebraic group schemes (not necessarily affine) over a field. After defining them and giving some examples, we show that most of the basic theory of abstract groups (subgroups, normal subgroups, normalizers, centralizers, Noether isomorphism theorems, subnormal series, etc.) carries over with little change to algebraic group schemes. We relate affine algebraic group schemes to Hopf algebras, and we prove that all algebraic group schemes in characteristic zero are smooth. We study the linear representations of algebraic group schemes and their actions on algebraic schemes. We show that every algebraic group scheme is an extension of an étale group scheme by a connected algebraic group scheme, and that every smooth connected group scheme over a perfect field is an extension of an abelian variety by an affine group scheme (Barsotti–Chevalley theorem).

Beginning with Chapter 9, all group schemes are affine.

B. Preliminaries on affine algebraic groups (Chapters 9–11)

The next three chapters are preliminary to the more detailed study of affine algebraic group schemes in the later chapters. They cover basic Tannakian theory, in which the category of representations of an algebraic group scheme plays the role of the topological dual of a locally compact abelian group, Jordan decompositions, the Lie algebra of an algebraic group, and the structure of finite group schemes. Throughout this work we emphasize the Tannakian point of view in which the group and its category of representations are placed on an equal footing.

C. Solvable affine algebraic groups (Chapters 12–16)

The next five chapters study solvable algebraic group schemes. Among these are the diagonalizable groups, the unipotent groups, and the trigonalizable groups.

An algebraic group G is diagonalizable if every linear representation of G is a direct sum of one-dimensional representations; in other words if, relative to some basis, the image of G lies in the algebraic subgroup of diagonal matrices in GL_n. An algebraic group that becomes diagonalizable over an extension of the base field is said to be of multiplicative type.

An algebraic group G is unipotent if every nonzero representation of G contains a nonzero fixed vector. This implies that every representation has a basis for which the image of G lies in the algebraic subgroup of strictly upper triangular matrices in GL_n.

An algebraic group G is trigonalizable if every simple representation has dimension one. This implies that every representation has a basis for which the image of G lies in the algebraic subgroup of upper triangular matrices in GL_n. The trigonalizable groups are exactly the extensions of diagonalizable groups by unipotent groups. Trigonalizable groups are solvable, and the Lie–Kolchin theorem says that all smooth connected solvable algebraic groups become trigonalizable over a finite extension of the base field.

D. Reductive algebraic groups (Chapters 17–25)

This is the heart of the book, The first seven chapters develop in detail the structure theory of split reductive groups and their representations in terms of their root data. Chapter 24 exhibits all the almost-simple algebraic groups, and Chapter 25 explains how the theory of split groups extends to the nonsplit case.

E. Appendices

The first appendix reviews the definitions and statements from algebraic geometry needed in the book. Experts need only note that, as we always work with schemes of finite type over a base field k, it is natural to ignore the nonclosed points (which we do).

The second appendix proves the existence of a quotient of an algebraic group by an algebraic subgroup. This is an important result, but the existence of nilpotents makes the proof difficult, and so most readers should simply accept the statement.

The third appendix reviews the combinatorial objects, root systems and root data, on which the theory of split reductive groups is based.

History

Apart from occasional brief remarks, we ignore the history of the subject, which is quite complex. Many major results were discovered in one situation, and then extended to other more general situations, sometimes easily and sometimes only with difficulty. Without too much exaggeration, one can say that all the theory of algebraic group schemes does is show that the theory of Killing and Cartan for "local" objects over $\mathbb{C}$ extends in a natural way to "global" objects over arbitrary fields.

Conventions and notation

Throughout, k is a field and R is a finitely generated k-algebra.[1] All k-algebras and R-algebras are required to be commutative and finitely generated unless it is specified otherwise. Noncommutative algebras are referred to as "algebras over k" rather than "k-algebras". Unadorned tensor products are over k. An extension of k is a field containing k, and a separable extension is a separable algebraic extension. When V is a vector space over k, we often write V_R for $V \otimes R$; for $v \in V$, we let $v_R = v \otimes 1 \in V_R$. The symbol k^{a} denotes an algebraic closure of k and k^{s} (resp. k^{i}) denotes the separable (resp. perfect) closure of k in k^{a}. The characteristic exponent of k is p or 1 according as its characteristic is p or 0. The group of invertible elements of a ring R is denoted by $R^{\times}$. The symbol Alg_R denotes the category of finitely generated R-algebras.

An algebraic scheme over k (or algebraic k-scheme) is a scheme of finite type over k. An algebraic scheme is an algebraic variety if it is geometrically reduced and separated. By a "point" of an algebraic scheme or variety over k we always mean a closed point. For an algebraic scheme $(X, \mathcal{O}_X)$ over k, we usually let X denote the scheme and $|X|$ the underlying topological space of closed points. For a locally closed subset Z of $|X|$ (resp. subscheme Z of X), the reduced subscheme of X with underlying space Z (resp. $|Z|$) is denoted by Z_{red}. The residue field at a point x of X is denoted by $\kappa(x)$. When the base field k is understood, we omit it, and write "algebraic scheme" for "algebraic scheme over k". Unadorned products of algebraic k-schemes are over k. See Appendix A for more details.

We let $\mathbb{Z}$ denote the ring of integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, and $\mathbb{F}_p$ the field of p elements (p prime).

A functor is said to be an equivalence of categories if it is fully faithful and essentially surjective. A sufficiently strong version of the axiom of global choice then implies that there exists a quasi-inverse to the functor. We sometimes loosely refer to a natural transformation of functors as a map of functors.

All categories are locally small (i.e., the morphisms from one object to a second are required to form a set). When the objects form a set, the category is said to be small. A category is essentially small if it is equivalent to a small subcategory.

Let P be a partially ordered set. A greatest element of P is a $g \in P$ such that $a \le g$ for all $a \in P$. An element m in P is maximal if $m \le a$ implies $a = m$. A greatest element is a unique maximal element. Least and minimal elements are defined similarly. When the partial order is inclusion, we replace least and greatest with smallest and largest. We sometimes use $[x]$ to denote the class of x under an equivalence relation.

[1] Except in Appendix C, where R is a set of roots.

Following Bourbaki, we let $\mathbb{N} = \{0, 1, 2, \ldots\}$. An integer is positive if it lies in $\mathbb{N}$. A set with an associative binary operation is a semigroup. A monoid is a semigroup with a neutral element.

By $A \simeq B$ we mean that A and B are canonically isomorphic (or that there is a given or unique isomorphism), and by $A \approx B$ we mean simply that A and B are isomorphic (there exists an isomorphism). The notation $A \subset B$ means that A is a subset of B (not necessarily proper). A diagram $A \to B \rightrightarrows C$ is exact if the first arrow is the equalizer of the pair of arrows.

Suppose that p and q are statements depending on a field k and we wish to prove that $p(k)$ implies $q(k)$. If $p(k)$ implies $p(k^{\mathrm{a}})$ and $q(k^{\mathrm{a}})$ implies $q(k)$, then it suffices to prove that $p(k^{\mathrm{a}})$ implies $q(k^{\mathrm{a}})$. In such a situation, we simply say that "we may suppose that k is algebraically closed".

We often omit "algebraic" from such expressions as "algebraic subgroup", "unipotent algebraic group", and "semisimple algebraic group". After p. 162, all algebraic groups are affine.

We use the terminology of modern (post 1960) algebraic geometry; for example, for algebraic groups over a field k, a homomorphism is automatically defined over k, not over some large algebraically closed field.[2]

Throughout, "algebraic group scheme" is shortened to "algebraic group". A statement here may be stronger than a statement in Borel 1991 or Springer 1998 even when the two are word for word the same.[3]

All constructions are to be understood as being in the sense of schemes. For example, fibres of maps of algebraic varieties need not be reduced, and the kernel of a homomorphism of smooth algebraic groups need not be smooth.

Numbering

A reference "17.56" is to item 56 of Chapter 17. A reference "(112)" is to the 112th numbered equation in the book (we include the page number where necessary). Section 17c is Section c of Chapter 17 and Section Ac is Section c of Appendix A. The exercises in Chapter 17 are numbered 17-1, 17-2, ...

Foundations

We use the von Neumann–Bernays–Gödel (NBG) set theory with the axiom of choice, which is a conservative extension of Zermelo–Fraenkel set theory with the axiom of choice (ZFC). This means that a sentence that does not quantify over a proper class is a theorem of NBG if and only if it is a theorem of ZFC. The advantage of NBG is that it allows us to speak of classes.

It is not possible to define an "unlimited category theory" that includes the category of *all* sets, the category of *all* groups, etc., and also the categories of

[2] As much as possible, our statements make sense in a world without choice, where algebraic closures need not exist.

[3] An example is Chevalley's theorem on representations; see 4.30.

functors from one of these categories to another. The category of functors from the category Alg_k of *all* finitely generated k-algebras to groups is not locally small. Instead, we should consider the functors from a subcategory Alg_k^0 whose objects are small in some sense. For example, fix a family of symbols $(T_i)_{i\in\mathbb{N}}$ indexed by $\mathbb{N}$, and let Alg_k^0 denote the category of k-algebras of the form $k[T_0,\ldots,T_n]/\mathfrak{a}$ for some $n\in\mathbb{N}$ and ideal $\mathfrak{a}$ in $k[T_0,\ldots,T_n]$. Then the objects of Alg_k^0 are indexed by the ideals in some subring $k[T_0,\ldots,T_n]$ of $k[T_0,\ldots]$ – in particular, they form a set, and so Alg_k^0 is small. The inclusion functor $\mathsf{Alg}_k^0 \hookrightarrow \mathsf{Alg}_k$ is an equivalence of categories. Choosing a quasi-inverse amounts to choosing an ordered set of generators for each finitely generated k-algebra. Once a quasi-inverse has been chosen, every functor on Alg_k^0 has a well-defined extension to Alg_k.

Readers willing to assume additional axioms in set theory may use Mac Lane's "one-universe" solution to defining functor categories (Mac Lane 1969) or Grothendieck's "multi-universe" solution (DG, p. xv), and define Alg_k^0 to consist of the k-algebras that are small relative to the chosen universe.

In the text, we ignore these questions.

Prerequisites

A first course in algebraic geometry (including basic commutative algebra). Since these vary greatly, we review the definitions and statements that we need from algebraic geometry in Appendix A. In a few proofs, which can be skipped, we assume somewhat more.

References

The citations are author–year, except for the following abbreviations:

CA = Milne 2017 (A Primer of Commutative Algebra).

DG = Demazure and Gabriel 1970 (*Groupes algébriques*).

EGA = Grothendieck 1967 (*Eléments de géometrie algébrique*).

SGA 3 = Demazure and Grothendieck 2011 (*Schémas en groupes*).

SHS = Demazure et al. 1966 (*Séminaire Heidelberg–Strasbourg 1965–66*).

CHAPTER 1

Definitions and Basic Properties

Recall that k is a field, and that an algebraic k-scheme is a scheme of finite type over k. We let $* = \operatorname{Spm}(k)$.

a. Definition

An algebraic group over k is a group object in the category of algebraic schemes over k. In detail, this means the following.

DEFINITION 1.1. Let G be an algebraic scheme over k, and let $m\colon G \times G \to G$ be a morphism. The pair (G, m) is an ***algebraic group*** over k if there exist morphisms

$$e\colon * \to G, \quad \operatorname{inv}\colon G \to G,$$

such that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mathrm{id}\times m} & G\times G \\ \downarrow{\scriptstyle m\times\mathrm{id}} & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccccc} *\times G & \xrightarrow{e\times\mathrm{id}} & G\times G & \xleftarrow{\mathrm{id}\times e} & G\times * \\ & \searrow{\scriptstyle\simeq} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\simeq} & \\ & & G & & \end{array} \tag{1}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{id})} & G\times G & \xleftarrow{(\mathrm{id},\mathrm{inv})} & G \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \xrightarrow{e} & G & \xleftarrow{e} & * \end{array} \tag{2}$$

When G is a variety, we call (G, m) a ***group variety***, and when G is an affine scheme, we call (G, m) an ***affine algebraic group***.

For example,

$$\mathrm{SL}_n \stackrel{\text{def}}{=} \operatorname{Spm} k[T_{11}, T_{12}, \ldots, T_{nn}]/(\det(T_{ij}) - 1)$$

becomes an affine group variety with the usual matrix multiplication on points. For many more examples, see Chapter 2.

Similarly, an ***algebraic monoid*** over k is an algebraic scheme M over k together with morphisms $m: M \times M \to M$ and $e: * \to M$ such that the diagrams (1) commute.

DEFINITION 1.2. A ***homomorphism*** $\varphi: (G,m) \to (G',m')$ of algebraic groups is a morphism $\varphi: G \to G'$ of algebraic schemes such that $\varphi \circ m = m' \circ (\varphi \times \varphi)$.

An algebraic group G is ***trivial*** if $e: * \to G$ is an isomorphism, and a homomorphism $G \to G'$ is ***trivial*** if it factors through $e': * \to G'$. We often write e for the trivial algebraic group.

DEFINITION 1.3. An ***algebraic subgroup*** of an algebraic group (G, m_G) over k is an algebraic group (H, m_H) over k such that H is a k-subscheme of G and the inclusion map is a homomorphism of algebraic groups. An algebraic subgroup is called a ***subgroup variety*** if its underlying scheme is a variety.

Let (G, m_G) be an algebraic group and H a nonempty subscheme of G. If $m_G|H \times H$ and $\mathrm{inv}_G|H$ factor through H, then $(H, m_G|H \times H)$ is an algebraic subgroup of G.

Let (G,m) be an algebraic group over k. For any field k' containing k, the pair $(G_{k'}, m_{k'})$ is an algebraic group over k', said to have been obtained from (G,m) by ***extension of scalars*** or ***extension of the base field***.

Algebraic groups as functors

The K-points of an algebraic scheme X with K a field do not see the nilpotents in the structure sheaf. Thus, we are led to consider the R-points with R a k-algebra. Once we do that, the points capture *all* information about X.

1.4. An algebraic scheme X over k defines a functor

$$\tilde{X}: \mathsf{Alg}_k \to \mathsf{Set}, \quad R \rightsquigarrow X(R).$$

For example, if X is affine, say, $X = \mathrm{Spm}(A)$, then

$$X(R) = \mathrm{Hom}_{k\text{-algebra}}(A, R).$$

The functor $X \rightsquigarrow \tilde{X}$ is fully faithful (Yoneda lemma, A.33); in particular, $\tilde{X}$ determines X uniquely up to a unique isomorphism. We say that a functor from k-algebras to sets is representable if it is of the form $\tilde{X}$ for an algebraic scheme X over k.

If (G,m) is an algebraic group over k, then $R \rightsquigarrow (G(R), m(R))$ is a functor from k-algebras to groups.

Let X be an algebraic scheme over k, and suppose that we are given a factorization of $\tilde{X}$ through the category of groups. Then the maps

$$x, y \mapsto xy: X(R) \times X(R) \to X(R), \;\; * \mapsto e: * \to X(R), \;\; x \mapsto x^{-1}: X(R) \to X(R)$$

given by the group structures on the sets $X(R)$ define, by the Yoneda lemma, morphisms

$$m\colon X\times X\to X,\ *\to X,\ \mathrm{inv}\colon X\to X$$

making the diagrams (1) and (2) commute. Therefore, (X,m) is an algebraic group over k.

Combining these two statements, we see that to give an algebraic group over k amounts to giving a functor $\mathsf{Alg}_k\to\mathsf{Grp}$ whose underlying functor to sets is representable by an algebraic scheme. We write $\tilde{G}$ for G regarded as a functor to groups.

From this perspective, SL_n can be described as the algebraic group over k sending R to the group $\mathrm{SL}_n(R)$ of $n\times n$ matrices with entries in R and determinant 1.

The functor $R\rightsquigarrow(R,+)$ is represented by $\mathrm{Spm}(k[T])$, and hence is an algebraic group $\mathbb{G}_a$. Similarly, the functor $R\rightsquigarrow(R^\times,\times)$ is represented by $\mathrm{Spm}(k[T,T^{-1}])$, and hence is an algebraic group $\mathbb{G}_m$. See 2.1 and 2.2 below.

We often describe a homomorphism of algebraic groups by giving its action on R-points. For example, when we say that $\mathrm{inv}\colon G\to G$ is the map $x\mapsto x^{-1}$, we mean that, for all k-algebras R and all $x\in G(R)$, $\mathrm{inv}(x)=x^{-1}$.

1.5. If (H,m_H) is an algebraic subgroup of (G,m_G), then $H(R)$ is a subgroup of $G(R)$ for all k-algebras R. Conversely, if H is an algebraic subscheme of G such that $H(R)$ is a subgroup of $G(R)$ for all k-algebras R, then the Yoneda lemma (A.33) shows that the maps

$$(h,h')\mapsto hh'\colon H(R)\times H(R)\to H(R)$$

arise from a morphism $m_H\colon H\times H\to H$ and that (H,m_H) is an algebraic subgroup of (G,m_G).

1.6. Consider the functor of k-algebras $\mu_3\colon R\rightsquigarrow\{a\in R\mid a^3=1\}$. This is represented by $\mathrm{Spm}(k[T]/(T^3-1))$, and so it is an algebraic group. We consider three cases.

(a) The field k is algebraically closed of characteristic $\neq 3$. Then

$$k[T]/(T^3-1)\simeq k[T]/(T-1)\times k[T]/(T-\zeta)\times k[T]/(T-\zeta^2)$$

where $1,\zeta,\zeta^2$ are the cube roots of 1 in k. Thus, μ_3 is a disjoint union of three copies of $\mathrm{Spm}(k)$ indexed by the cube roots of 1 in k.

(b) The field k is of characteristic $\neq 3$ but does not contain a primitive cube root of 1. Then

$$k[T]/(T^3-1)\simeq k[T]/(T-1)\times k[T]/(T^2+T+1),$$

and so μ_3 is a disjoint union of $\mathrm{Spm}(k)$ and $\mathrm{Spm}(k[\zeta])$ where ζ is a primitive cube root of 1 in k^{s}.

(c) The field k is of characteristic 3. Then $T^3-1=(T-1)^3$, and so μ_3 is not reduced. Although $\mu_3(K)=1$ for all fields K containing k, the algebraic group μ_3 is not trivial. Certainly, $\mu_3(R)$ may be nonzero if R has nilpotents.

Homogeneity

Recall that, for an algebraic scheme X over k, we write $|X|$ for the underlying topological space of X, and $\kappa(x)$ for the residue field at a point x of $|X|$ (it is a finite extension of k). We can identify $X(k)$ with the set of points x of $|X|$ such that $\kappa(x) = k$ (CA 13.4). An algebraic scheme X over k is said to be ***homogeneous*** if the group of automorphisms of X (as a k-scheme) acts transitively on $|X|$. We shall see that algebraic groups are homogeneous when k is algebraically closed

1.7. Let (G,m) be an algebraic group over k. The map $m(k): G(k) \times G(k) \to G(k)$ makes $G(k)$ into a group with neutral element $e(*)$ and inverse map $\mathrm{inv}(k)$.

When k is algebraically closed, $G(k) = |G|$, and so $m: G \times G \to G$ makes $|G|$ into a group. The maps $x \mapsto x^{-1}$ and $x \mapsto ax$ ($a \in G(k)$) are automorphisms of $|G|$ as a topological space.

In general, when k is not algebraically closed, m does not make $|G|$ into a group, and even when k is algebraically closed, it does not make $|G|$ into a *topological* group.

1.8. Let (G,m) be an algebraic group over k. For each $a \in G(k)$, there is a translation map

$$l_a: G \simeq \{a\} \times G \xrightarrow{m} G, \quad x \mapsto ax.$$

For $a, b \in G(k)$,

$$l_a \circ l_b = l_{ab}$$

and $l_e = \mathrm{id}$. Therefore $l_a \circ l_{a^{-1}} = \mathrm{id} = l_{a^{-1}} \circ l_a$, and so l_a is an isomorphism sending e to a. Hence G is homogeneous when k is algebraically closed (but not in general otherwise; see 1.6(b)).

Density of points

Because we allow nilpotents in the structure sheaf, a morphism $X \to Y$ of algebraic schemes is not in general determined by its effect on $X(k)$, even when k is algebraically closed. We introduce some terminology to handle this.

DEFINITION 1.9. Let X be an algebraic scheme over k and S a subset of $X(k)$. We say that S is ***schematically dense*** in X if the only closed subscheme Z of X such that $S \subset Z(k)$ is X itself.

Let $X = \mathrm{Spm}(A)$, and let S be a subset of $X(k)$. Let $Z = \mathrm{Spm}(A/\mathfrak{a})$ be a closed subscheme of X. Then $S \subset Z(k)$ if and only if $\mathfrak{a} \subset \mathfrak{m}$ for all $\mathfrak{m} \in S$. Therefore, S is schematically dense in X if and only if $\bigcap \{\mathfrak{m} \mid \mathfrak{m} \in S\} = 0$.

PROPOSITION 1.10. *Let X be an algebraic scheme over k and S a subset of $X(k) \subset |X|$. The following conditions are equivalent:*

(a) *S is schematically dense in X;*

(b) *X is reduced and S is dense in $|X|$;*

(c) *the family of homomorphisms*

$$f \mapsto f(s)\colon \mathcal{O}_X \to \kappa(s) = k, \quad s \in S,$$

is injective.

PROOF. (a)⇒(b). Let $\bar{S}$ denote the closure of S in $|X|$. There is a unique reduced subscheme Z of X with underlying space $\bar{S}$. As $S \subset |Z|$, the scheme $Z = X$, and so X is reduced with underlying space $\bar{S}$.

(b)⇒(c). Let U be an open affine subscheme of X, and let $A = \mathcal{O}_X(U)$. Let $f \in A$ be such that $f(s) = 0$ for all $s \in S \cap |U|$. Then $f(u) = 0$ for all $u \in |U|$ because $S \cap |U|$ is dense in $|U|$. This means that f lies in all maximal ideals of A, and therefore lies in the radical of A, which is zero because X is reduced (CA 13.11).

(c)⇒(a). Let Z be a closed subscheme of X such that $S \subset Z(k)$. Because Z is closed in X, the homomorphism $\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. Because $S \subset Z(k)$, the maps $f \mapsto f(s)\colon \mathcal{O}_X \to \kappa(s)$, $s \in S$, factor through $\mathcal{O}_Z$, and so $\mathcal{O}_X \to \mathcal{O}_Z$ is injective, hence an isomorphism, which implies that $Z = X$. □

PROPOSITION 1.11. *A schematically dense subset remains schematically dense under extension of the base field.*

PROOF. Let k' be a field containing k, and let $S \subset X(k)$ be schematically dense in X. We may suppose that X is affine, say, $X = \mathrm{Spm}(A)$. Let $s'\colon A \otimes k' \to k'$ be the map obtained from $s\colon A \to \kappa(s) = k$ by extension of scalars. The family s', $s \in S$, is injective because the family s, $s \in S$, is injective and k' is flat over k. □

COROLLARY 1.12. *If X admits a schematically dense subset $S \subset X(k)$, then it is geometrically reduced.*

PROOF. When regarded as a subset of $X(k^{\mathrm{a}})$, S is schematically dense in $X_{k^{\mathrm{a}}}$, which is therefore reduced. □

PROPOSITION 1.13. *Let $u, v\colon X \rightrightarrows Y$ be morphisms from X to a separated algebraic scheme Y over k. If S is schematically dense in X and $u(s) = v(s)$ for all $s \in S$, then $u = v$.*

PROOF. Because Y is separated, the equalizer of the pair of maps is closed in X. As its underlying space contains S, it equals X. □

REMARK 1.14. Some of the above discussion extends to base rings. For example, let X be an algebraic scheme over a field k and let S be a schematically dense subset of $X(k)$. Let R be a k-algebra and, for $s \in S$, let

$$s' = s \times_{\mathrm{Spm}(k)} \mathrm{Spm}(R) \subset X' = X \times_{\mathrm{Spm}(k)} (R).$$

As in the proof of 1.11, the family of maps $\mathcal{O}_{X'} \to \mathcal{O}_{s'}(s') = R$ is injective. It follows, as in the proof of 1.10, that the only closed R-subscheme of X' containing all s' is X' itself.