Bond Pricing and Yield-Curve Modelling

A Structural Approach

This book provides the theoretical foundations (no-arbitrage, convexity, expectations, affine modelling) for a treatment of government bond markets; presents and critically discusses the wealth of empirical findings that have appeared in the literature in the last decade; and introduces the ‘structural’ models that are used by central banks, institutional investors, sovereign wealth funds, academics and advanced practitioners to model the yield curve, to answer policy questions, to estimate the magnitude of the risk premium, to gauge market expectations and to assess investment opportunities. The book weaves precise theory with up-to-date empirical evidence to build, with the minimum mathematical sophistication required for the task, a critical understanding of what drives the government bond market.

Riccardo Rebonato is Professor of Finance at the EDHEC Business School and the EDHEC Risk Institute and holds the EDHEC PIMCO Research Chair. He has been Global Head of Fixed Income and FX Analytics at PIMCO and Head of Research, Risk Management and Derivatives Trading at several major international banks. He has previously held academic positions at Imperial College and Oxford University, and has been a board director for the International Swaps and Derivatives Association (ISDA) and the Global Association of Risk Professionals (GARP). He currently sits on the board of The Nine Dots Prize. He is the author of several books and articles in finance and risk management, including *Portfolio Management under Stress* (2004).
Bond Pricing and
Yield-Curve Modelling
A Structural Approach

Riccardo Rebonato
EDHEC Business School
EDHEC Risk Institute
To the memory of my father, to my wife and to my son, with thanks.
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Symbols and Abbreviations

LATIN SYMBOLS

\( A_t \) = A scalar that enters the expression for \( P_T^t \), ie, for the price at time \( t \) of a zero-coupon bond of maturity \( T \), as in \( P_T^t = e^{A_T^t + (B_T^t)^T x_t} \)

\( B_t \) = A vector that enters the expression for \( P_T^t \), ie, for the price at time \( t \) of a zero-coupon bond of maturity \( T \), as in \( P_T^t = e^{A_T^t + (B_T^t)^T x_t} \)

\( BEI_{T, t} \) = Break-even inflation at time \( t \) for maturity \( T \)

\( c_t \) = Consumption at time \( t \)

\( Conv_T^t \) = Time-\( t \) convexity of a \( T \)-maturity (discount) bond

\( D = \sum_{i=1,n} a_i (B_i^t) (B_i^t)^T \) = Time-\( t \) duration of a \( T \)-maturity (discount) bond

\( E[I]_T^t \) = Time-\( t \) expected inflation per unit time over the period \([t, T]\)

\( f_T^t \) = Value at time \( t \) of an instantaneous forward rate expiring at time \( T \)

\( F_{T, T+\tau}^t \) = Value at time \( t \) of a discrete forward rate expiring at time \( T \) and covering a notional borrowing/lending period of length \( \tau \)

\( \{l_i\} = \) Eigenvalues of the reversion-speed matrix

\( m_{t+1} \) = Stochastic discount factor for time \( t + 1 \)

\( M_t \) = \( \beta u (c_t) \) stochastic discount factor in continuous time

\( p_{T_t}^t \) = Log price at time \( t \) of a zero-coupon bond of maturity \( T \)

\( p_s \) = Probability of reaching state \( s \)

\( P_T^t \) = Price at time \( t \) of a zero-coupon bond of maturity \( T \)

\( P_{t, T}^n \) = Price at time \( t \) of a real zero-coupon bond of maturity \( T \)

\( q_t = e^Q_t \)

\( Q_t \) = Value of the price process at time \( t \)

\( r_t \) = Value of the short rate at time \( t \)

\( r_f^t \) = Value of the real short rate at time \( t \)

\( r_f \) = Riskless rate

\( ret_T^t \) = Annualized return from investing in the \( T \)-maturity discount bond

\( R_f^t \) = Gross one-period riskless return

\( S \) = Volatility matrix for an affine model

\( S_t \) = Time-\( t \) price of a generic security
Symbols and Abbreviations

- \( sp_s \) = State price at time \( s \)
- \( SR \) = Sharpe Ratio
- \( u(c_t) \) = Utility for consumption at time \( t \)
- \( W \) = Initial wealth
- \( x_t \) = A scalar or a vector that denotes the state variables in an affine model
- \( xret_t \) = Annualized excess return from investing in the \( T \)-maturity discount bond
- \( y_T^r \) = Yield at time \( t \) of a zero-coupon bond of maturity \( T \)
- \( y_{\text{nom},T}^r \) = Nominal yield at time \( t \) of a zero-coupon bond of maturity \( T \)
- \( y_{\text{real},T}^r \) = Real yield at time \( t \) of a zero-coupon bond of maturity \( T \)

Greek Symbols

- \( \alpha_t^T \) = \(- \frac{1}{T-t} A_{t,T}^T \): One of the two quantities that links yields to state variables for affine models, as in \( y_t^r = \alpha_t^T + (\beta_t^T)^T x_t \)
- \( \beta_t^T \) = \(- \frac{1}{T-t} A_{t,T}^T \): One of the two quantities that links yields to state variables for affine models, as in \( y_t^r = \alpha_t^T + (\beta_t^T)^T x_t \)
- \( \beta \) = Time impatience term in the stochastic discount factor: \( m_{t+1} = \beta \frac{u'(G_{t+1})}{u'(C_t)} \)
- \( \kappa \) = Reversion speed in a one-dimensional mean-reverting process
- \( \theta \) = Reversion level (vector or scalar) for an affine model
- \( \Theta_t^T \equiv \left( \frac{\theta_t^T}{x_t} \right)^{-1} \)
- \( \lambda \) = Market price of risk
- \( \Lambda \) = Matrix of eigenvalues of the reversion-speed matrix
- \( \mu_t^{P(Q)} \) = Drift in the real-world (risk-neutral) measure of the short rate
- \( v_0 \) = The price of a security today
- \( \pi_s \) = State-price deflator
- \( \pi_{\text{real},T} \) = Real state-price deflator
- \( \sigma_P \) = Bond price volatility
- \( \sigma_r \) = Volatility of the short rate
- \( \sigma_N \) = Volatility of a yield of expiry \( T \)
- \( \tau \equiv T - t \)
- \( \omega_i \) = The \( i \)th weight in a portfolio

Fraktur and Calligraphic Symbols

- \( \mathfrak{C} \) = Convexity of a portfolio
- \( \mathfrak{C}_P^T \text{(Vasicek)} \) = Bond price convexity in the Vasicek model
- \( \mathfrak{C}_y^T \text{(Vasicek)} \) = Yield convexity in the Vasicek model
- \( Ci_{t,T}^r \) = Convexity term in the expression for break-even inflation in the real-world measure
- \( Ci_{t,T}^Q \) = Convexity term in the expression for break-even inflation in the risk-neutral measure
Symbols and Abbreviations

\( C_{i,T} \) = Convexity term in the expression for break-even inflation in the risk-neutral measure

\( K \) = Reversion speed matrix for an affine model

\( L_{T} \) = Liquidity component in the decomposition of break-even inflation:

\( BEI_{T} = \mathbb{E} \left( R_{T}^{T} \right)^{P} + P_{T}^{T} + L_{T} \)

\( M_{N}^{T} \) = The Ratio of the nominal stochastic discount factors: \( M_{N}^{T} = \frac{\pi^{N}(T)}{\pi^{N}(0)} \)

\( N(\mu, \sigma^{2}) \) = Normal distribution with mean, \( \mu \), and variance, \( \sigma^{2} \)

\( P_{T}^{T} \) = Risk premium required by an investor in order to bear inflation risk

\( \mathbb{P} \) = Real-world measure

\( Q \) = Risk-neutral measure

\( S \) = Subjective measure

\( S_{\tau} \) = The Set of all the possible future states at time \( \tau \)

\( T \) = Terminal measure