

Abstract Algebra with Applications

Abstract Algebra with Applications provides a friendly and concise introduction to algebra, with an emphasis on its uses in the modern world. The first part of this book covers groups, after some preliminaries on sets, functions, relations, and induction, and features applications such as public-key cryptography, Sudoku, the finite Fourier transform, and symmetry in chemistry and physics. The second part of this book covers rings and fields, and features applications such as random number generators, error-correcting codes, the Google page rank algorithm, communication networks, and elliptic curve cryptography.

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Abstract Algebra with Applications

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To the bears and koalas

Contents

<i>List of Figures</i>	ix
<i>Preface</i>	xiii

PART I GROUPS

1 Preliminaries	3
1.1 Introduction	3
1.2 Sets	6
1.3 The Integers	9
1.4 Mathematical Induction	14
1.5 Divisibility, Greatest Common Divisor, Primes, and Unique Factorization	19
1.6 Modular Arithmetic, Congruences	26
1.7 Relations	30
1.8 Functions, the Pigeonhole Principle, and Binary Operations	34
2 Groups: A Beginning	43
2.1 What is a Group?	43
2.2 Visualizing Groups	52
2.3 More Examples of Groups and Some Basic Facts	56
2.4 Subgroups	64
2.5 Cyclic Groups are Our Friends	72
3 Groups: There's More	81
3.1 Groups of Permutations	81
3.2 Isomorphisms and Cayley's Theorem	89
3.3 Cosets, Lagrange's Theorem, and Normal Subgroups	93
3.4 Building New Groups from Old, I: Quotient or Factor Groups G/H	98
3.5 Group Homomorphism	102
3.6 Building New Groups from Old, II: Direct Product of Groups	108
3.7 Group Actions	114
4 Applications and More Examples of Groups	124
4.1 Public-Key Cryptography	124
4.2 Chemistry and the Finite Fourier Transform	129
4.3 Groups and Conservation Laws in Physics	135
4.4 Puzzles	142
4.5 Small Groups	146

PART II RINGS

5	Rings: A Beginning	157
5.1	Introduction	157
5.2	What is a Ring?	158
5.3	Integral Domains and Fields are Nicer Rings	166
5.4	Building New Rings from Old: Quotients and Direct Sums of Rings	173
5.5	Polynomial Rings	180
5.6	Quotients of Polynomial Rings	185
6	Rings: There's More	189
6.1	Ring Homomorphisms	189
6.2	The Chinese Remainder Theorem	193
6.3	More Stories about $F[x]$ Including Comparisons with \mathbb{Z}	198
6.4	Field of Fractions or Quotients	202
7	Vector Spaces and Finite Fields	206
7.1	Matrices and Vector Spaces over Arbitrary Fields and Rings like \mathbb{Z}	206
7.2	Linear Functions or Mappings	218
7.3	Determinants	224
7.4	Extension Fields: Algebraic versus Transcendental	229
7.5	Subfields and Field Extensions of Finite Fields	233
7.6	Galois Theory for Finite Fields	239
8	Applications of Rings	244
8.1	Random Number Generators	244
8.2	Error-Correcting Codes	256
8.3	Finite Upper Half Planes and Ramanujan Graphs	265
8.4	Eigenvalues, Random Walks on Graphs, and Google	272
8.5	Elliptic Curve Cryptography	282
	<i>References</i>	299
	<i>Index</i>	305

Figures

0.1	Benzene C_6H_6	xiv
0.2	Photoshopped flower	xiv
0.3	Hibiscus in Kauai	xiv
0.4	Picture with symmetry coming from the action of 2×2 matrices with nonzero determinant and elements in a finite field with 11 elements	xv
1.1	Intersection and union of square A and heart B	7
1.2	Cartesian product $[0, 1] \times \{2\}$	8
1.3	$[0, 1]^3$	8
1.4	Graph representing the hypercube $[0, 1]^4$	9
1.5	Integers on the line – only even ones are labelled	13
1.6	The first principle of mathematical induction. A penguin surveys an infinite line of equally spaced dominos. If the n th domino is close enough to knock over the $(n + 1)$ th domino, then once the penguin knocks over the first domino, they should all fall over	14
1.7	Here is an attempt to picture the second mathematical induction principle in which we arrange dominos so that various numbers of dominos are needed to knock over the dominos to their left. In this picture step 1 would be for the penguin to knock over d_1 and d_2	17
1.8	A color is placed at the (m, n) entry of a 101×101 matrix according to the value of $\gcd(m, n)$. This is an ArrayPlot in Mathematica	25
1.9	Rolling up the integers modulo 3	27
1.10	Mathematica picture of the $x < y$ relation for the integers between 1 and 50	31
1.11	Mathematica picture of the $y x$ relation for the integers between 1 and 50	31
1.12	Mathematica picture of the $x \equiv y \pmod{5}$ relation for the integers between 1 and 50	32
1.13	Poset diagram of the positive divisors of 24	34
1.14	The pigeonhole principle	39
2.1	The symmetries of a regular triangle are pictured	44
2.2	Part of a design with translational symmetry which should be imagined to stretch out to ∞ and $-\infty$	47
2.3	A figure with C_8 symmetry – not D_8 symmetry	48
2.4	Art from the Raja Ampat islands in the Indonesian part of New Guinea	48
2.5	Wallpaper from a Fourier series in two variables	50
2.6	Spherical wallpaper from spherical harmonics	51
2.7	Hyperbolic wallpaper from a modular form known as Δ on the upper half plane – a function with an invariance property under fractional linear transformation $(az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$	51

2.8	Group Explorer version of the multiplication table for C_6 , a cyclic group of order 6	52
2.9	Group Explorer version of the multiplication table for D_3 (alias S_3) with our upper case R and F replaced by lower case letters	52
2.10	Cayley graph of cyclic group $G = \langle a \rangle$ of order 6 with generating set $S = \{a\}$	53
2.11	Cayley graph of D_3 with generating set $\{R, F\}$. Our upper case letters are replaced by lower case in the diagram. If there are arrows in both directions on an edge, we omit the arrows	53
2.12	Undirected version of Cayley graph for $C_6 = \langle a \rangle$, generating set $S = \{a, a^{-1}\}$	53
2.13	Cayley graph of $C_6 = \langle a \rangle$, generating set $S = \{a, a^3, a^5\}$	53
2.14	Group Explorer version of the multiplication table for the Klein 4-group	54
2.15	Symmetrical designs	54
2.16	The Platonic solids	57
2.17	Poset diagram for subgroups of D_3 as defined in (2.8)	68
2.18	The Group Explorer version of the multiplication table for a cyclic group of order 10	73
2.19	Cayley graph $X(\langle a \rangle, \{a, a^{-1}\})$ for a cyclic group $\langle a \rangle$ of order 10	76
2.20	A less boring picture of a 10-cycle	76
2.21	Poset diagram of the subgroups of \mathbb{Z}_{24} under addition	78
2.22	Cycle diagram in the multiplicative group \mathbb{Z}_{15}^*	80
3.1	Tetrahedron	87
3.2	On the left is the Cayley graph for the Klein 4-group $K_4 = \{e, h, v, hv\}$, with generating set $S = \{h, v\}$ using the notation of Figure 2.14. On the right is the Schreier graph for K_4/H , where $H = \{e, h\}$, with the same set $S = \{h, v\}$	97
3.3	Roll up \mathbb{Z} to get $\mathbb{Z}/n\mathbb{Z}$	105
3.4	Roll up the real line to get a circle $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$	106
3.5	Cayley graph for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the generating set $\{(1, 0), (0, 1)\}$ and bears at the vertices	109
3.6	Cayley graph for $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the generating set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and koalas at the vertices	109
3.7	Cayley graph for $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the generating set $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$	110
3.8	A finite torus, which is the Cayley graph $X(\mathbb{Z}_{10} \oplus \mathbb{Z}_5, \{(\pm 1, 0), (0, \pm 1)\})$	111
3.9	The continuous torus (obtained from the plane modulo its integer points; i.e., $\mathbb{R} \oplus \mathbb{R}$ modulo $\mathbb{Z} \oplus \mathbb{Z}$)	111
3.10	Cayley graph for the quaternion group with generating set $\{\pm i, \pm j\}$	113
3.11	The 13 necklaces with six beads of two colors	121
3.12	The dodecahedron graph drawn by Mathematica	122
3.13	The cuboctahedron drawn by Mathematica	123
4.1	Vibrating system of two masses	142
4.2	Group Explorer's multiplication table for the semi-direct product $C_3 \rtimes C_4$	149
4.3	Group Explorer draws the Cayley graph $X(C_3 \rtimes C_4, \{a, b\})$	149
4.4	The Cayley graph $X(\text{Aff}(5), S_{1,2})$, with generating set defined by equation (4.16), has edges given by solid green lines while the dashed magenta lines are the edges of a dodecahedron	150

List of Figures

xi

4.5	Butterfly from Cayley graph of $\text{Heis}(\mathbb{Z}/169\mathbb{Z})$	151
4.6	A spanning tree for the tetrahedron graph is indicated in solid fuchsia lines. Since the three dashed purple edges are left out, the fundamental group of the tetrahedron graph is the free group on three generators. The arrows show a closed path on the tetrahedron graph	153
4.7	The bouquet of three loops obtained by collapsing the tree in the tetrahedron graph of Figure 4.6 to point a	153
4.8	A passion flower	154
5.1	The color at point $(x, y) \in \mathbb{Z}_{163}^2$ indicates the value of $x^2 + y^2 \pmod{163}$	157
5.2	Points (x, y) , for $x, y \in \mathbb{Z}_{11^2}, y \neq 0$, have the same color if $z = x + y\sqrt{\delta}$ are equivalent under the action of non-singular 2×2 matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entries in \mathbb{Z}_{11} . The action of g on z is by fractional linear transformation $z \rightarrow (az + b)/(cz + d) = gz$. Here δ is a fixed non-square in the field \mathbb{F}_{121} with 121 elements	158
5.3	Poset diagram of the ideals in \mathbb{Z}_{12}	178
5.4	A feedback shift register diagram corresponding to the finite field $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$ and the multiplication table given in the text	186
6.1	The Cayley graph $X(\mathbb{Z}_{15}, \{\pm 1 \pmod{15}\})$	196
6.2	The Cayley graph $X(\mathbb{Z}_{15}, \{5, 6, 9, 10 \pmod{15}\})$	196
7.1	The poset of subfields of $\mathbb{F}_{2^{24}}$	234
8.1	The Cayley graph $X(\mathbb{Z}_{17}^*, \{3 \pmod{17}\})$	245
8.2	The same graph as in Figure 8.1 except that now the vertices are given the usual ordering $1, 2, 3, 4, \dots, 16$	245
8.3	Plot of points $P_j = (j, v_j)$ whose second component is the real number $\frac{1}{499}$ times $7^j \pmod{499}$, identifying $7^j \pmod{499}$ as an integer between 1 and 498	248
8.4	Plot of points $P_j = (v_j, v_{j+1})$ whose first component is the real number $\frac{1}{499}$ times $7^j \pmod{499}$, identifying $7^j \pmod{499}$ as an integer between 1 and 498	248
8.5	Plot of points $P_j = (v_j, v_{j+1}, v_{j+2})$ whose first component is the real number $\frac{1}{499}$ times $7^j \pmod{499}$, identifying $7^j \pmod{499}$ as an integer between 1 and 498	249
8.6	Plot of points $P_j = (v_j, w_j)$ whose first component is the real number $\frac{1}{499}$ times $7^j \pmod{499}$, identifying $7^j \pmod{499}$ as an integer between 1 and 498 and whose second component is the analog with 499 replaced with 503	249
8.7	Points (v_i, w_i, z_i) from three vectors v, w, z formed from powers of generators of \mathbb{F}_p^* for $p = 499, 503$, and 521 , respectively	250
8.8	Feedback shift register corresponding to example 2	252
8.9	Sending a message of 0s and 1s to Professor Bolukxy on the planet Xotl	257
8.10	The matrix H_{32} where the 1s and -1 s have become red and purple	263
8.11	Color at point $z = x + y\sqrt{\delta}$ in H_{163} is found by computing the Poincaré distance $d(z, \sqrt{\delta})$	268
8.12	The graph on the left is $X_3(-1, 1)$, an octahedron, and that on the right is $X_5(2, 1)$ with the edges in green. The pink dashed lines on the right are the dodecahedron	269
8.13	Another version of Figure 5.2	270

8.14	A random walk on a pentagon. At time $t=0$, the big penguin is at vertex 1. At time $t=1$ the penguin has probability $\frac{1}{2}$ of being at vertex 2 and probability $\frac{1}{2}$ of being at vertex 5. So the penguins at these vertices are half size	275
8.15	Surfing a very small web	277
8.16	Real points (x, y) on the elliptic curve $y^2 = x^3 + x^2$	282
8.17	Real points (x, y) on the elliptic curve $y^2 = x^3 - x + 1$	283
8.18	Real points (x, y) on the elliptic curve $y^2 = x^3 - x$	283
8.19	Addition $A + B = C$ on the elliptic curve $y^2 = x^3 - x$ over \mathbb{R}	286
8.20	The rational points on the curve $y^2 + y = x^3 - x^2$ are a, b, c, d and the point at ∞	287
8.21	The pink squares indicate the points (x, y) on the elliptic curve $y^2 = x^3 - x + 1 \pmod{59}$. Points marked are: $A = (15, 36), B = (22, 40), C = (32, 46)$, with $A + B = -C = (32, 13)$	289
8.22	Level “curves” of $y^2 - x^3 - x + 1 \pmod{29}$	296
8.23	Smoothed level “curves” of $y^2 - x^3 - x + 1 \pmod{29}$	297
8.24	A photoshopped version of the level curves of $(y + 2x)^4 + (x - 2y)^4 \pmod{101}$	297

Preface

My goal for this book is to provide a friendly concise introduction to algebra with emphasis on its uses in the modern world – including a little history, concrete examples, and visualization. Beyond explaining the basics of the theory of groups, rings, and fields, I aim to give many answers to the question: What is it good for? The standard undergraduate mathematics course in the 1960s (when I was an undergraduate) proceeded from Definition 1.1.1 to Corollary 14.5.59 with little room for motivation, examples, history, and applications. I plan to stay as far as possible from that old format, modeling my discussion on G. Strang’s book [115], where the preface begins: “I believe that the teaching of linear algebra has become too abstract.” My feeling is that the teaching of modern algebra (the non-linear part) has become even more abstract. I will attempt to follow Strang’s lead and treat modern algebra in a way that will make sense to a large variety of students. On the other hand, the goal is to deal with some abstractions – groups, rings, and such things. Yes, it is abstraction and generalization that underlies the power of mathematics. Thus there will be some conflict between the applied and pure aspects of our subject.

The book is intended for a year-long undergraduate course in algebra. The intended audience is the less theoretically inclined undergraduates majoring in mathematics, the physical and social sciences, or engineering – including those in applied mathematics or those intending to get a teaching credential.

The prerequisites are minimal: comfort with the real numbers, the complex numbers, matrices, vector spaces at the level of calculus courses – and a bit of courage when asked to do a short proof.

In this age of computers, algebra may have replaced calculus (analysis) as the most important part of mathematics. For example:

1. Error-correcting codes are built into the DVD player and the computer. Who do you call to correct errors? The algebraist, that’s who!
2. Digital signal processing (such as that involved in medical scanners, weather prediction, the search for oil) is dominated by the fast Fourier transform or FFT. What is this? The FFT is a finite sum whose computation has been sped up considerably by an algebra trick which goes back to Gauss in 1805. Once more algebra, not analysis, rules.
3. In chemistry and physics, one studies structures with symmetry such as the benzene molecule (C_6H_6) depicted in Figure 0.1. What does the 6-fold symmetry have to do with the properties of benzene? Group theory is the tool one needs for this.
4. The search for secret codes – cryptography. Much of the modern world – particularly that which lives on the internet – depends on these codes being secure. But are they? We will consider public-key codes. And who do you call to figure out these codes? An algebraist!

5. The quest for beauty in art and nature. I would argue that symmetry groups are necessary tools for this quest. See Figures 0.2–0.4.

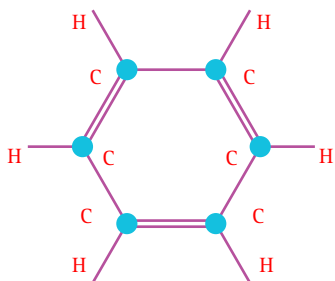


Figure 0.1 Benzene C_6H_6



Figure 0.2 Photostopped flower



Figure 0.3 Hibiscus in Kauai

Our goal here is to figure out enough group and ring theory to understand many of these applications. And we should note that both algebra and analysis are necessary for the applications. In fact, we shall see some limits, derivatives, and integrals before the last pages of this book. You can skip all the applications if you just want to learn the basics

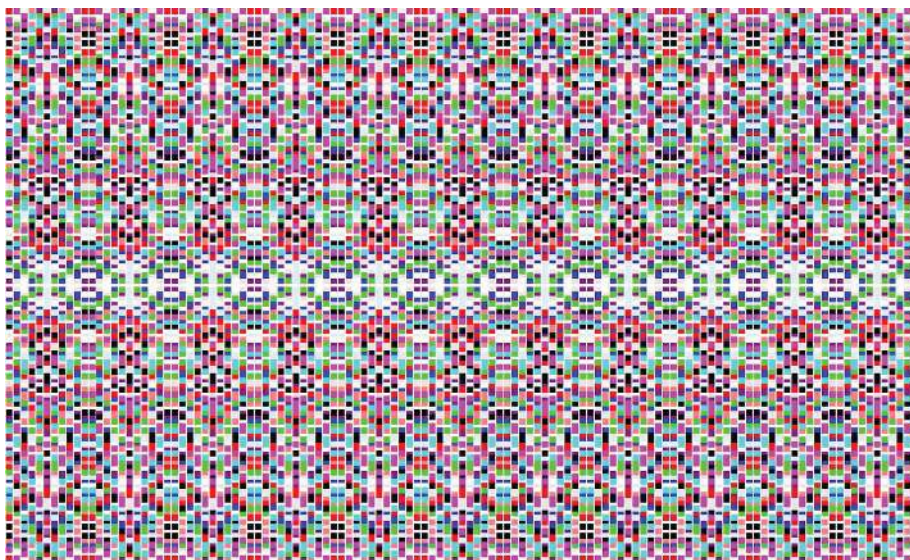


Figure 0.4 Picture with symmetry coming from the action of 2×2 matrices with nonzero determinant and elements in a finite field with 11 elements

of modern algebra. The non-applied sections are all independent of those on applications. However, you would be missing one of the big reasons that the subject is taught. And feel free to skip any applied sections you want to skip – or to add any missing application that you want to understand.

The first part of this book covers groups, after some preliminaries on sets, functions, relations, the integers, and mathematical induction. Of course every calculus student is familiar with the group of real numbers under addition and similarly with the group of nonzero real numbers under multiplication. We will consider many more examples – favorites being finite groups such as the group of symmetries of an equilateral triangle.

Much of our subject began with those favorite questions from high school algebra such as finding solutions to polynomial equations. It took methods of group theory to know when the solutions could not be found in terms only of n th roots. Galois, who died at age 21 in a duel in 1832, laid the foundations to answer such questions by looking at groups of permutations of the roots of a polynomial. These are now called Galois groups. See Edna E. Kramer [59, Chapter 16] or Ian Stewart [114] for some of the story of Galois and the history of algebra. Another reference for stories about Galois and the many people involved in the creation of this subject is *Men of Mathematics* by Eric Temple Bell [8]. This book is often criticized for lack of accuracy, but it is more exciting than most. I found it inspirational as an undergraduate – despite the title.

Another area that leads to our subject is number theory: the study of the ring of integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}.$$

The origins of this subject go back farther than Euclid's *Elements*. Euclid lived in Alexandria around 300 BC and his book covers more than the plane geometry we learned in high school. Much of the basic theory of the integers which we cover in Chapter 1 is to be found in Euclid's *Elements*. Why is it that the non-plane geometry part of Euclid's *Elements* does not seem to be taught in high school?

Polynomial equations with integer coefficients are often called Diophantine equations in honor of Diophantus who also lived in Alexandria, but much later (around AD 200). Yes, algebra is an old subject and one studied in many different countries. For example, the name “algebra” comes the word *al-jabr*, one of the two operations used to solve quadratic equations by the Persian mathematician and astronomer Mohamed ibn Musa al-Khwārizmī, who lived around AD 800.

A large part of this subject was created during many attempts to prove Fermat’s last theorem. This was a conjecture of Pierre de Fermat in 1637 stating that the equation $x^n + y^n = z^n$ can have no integer solutions x, y, z with $xyz \neq 0$ and $n > 2$. Fermat claimed to have a proof that did not fit in the margin of the book in which he wrote this conjecture. People attempted to prove this theorem without success until A. Wiles with the help of R. Taylor in 1995. People still seek an “elementary” proof.

Groups are sets with one operation satisfying the axioms to be listed in Section 2.1. After the basic definitions, we consider examples of small groups. We will visualize groups using Cayley graphs and various other diagrams such as Hasse or poset diagrams as well as cycle diagrams. Other topics of study include subgroups, cyclic groups, permutation groups, functions between groups preserving the group operations (homomorphisms), cosets of subgroups, building new groups whose elements are cosets of normal subgroups, direct products of groups, actions of groups on sets. We will consider such applications as public-key cryptography, the finite Fourier transform, and the chemistry of benzene. Favorite examples of groups include cyclic groups, permutation groups, symmetry groups of the regular polygons, matrix groups such as the Heisenberg group of 3×3 upper triangular matrices with real entries and 1 on the diagonal, the group operation being matrix multiplication.

The second part of this book covers rings and fields. Rings have two operations satisfying the axioms listed in Section 5.2. We denote the two operations as addition $+$ and multiplication $*$ or \cdot . The identity for addition is denoted 0. It is NOT assumed that multiplication is commutative: that is, it is not assumed that $ab = ba$. If multiplication is commutative, then the ring is called commutative. A field F is a commutative ring with an identity for multiplication (called $1 \neq 0$) such that the nonzero elements of F form a multiplicative group. Most of the rings considered here will be commutative. We will be particularly interested in finite fields like the field of integers modulo p , $\mathbb{Z}/p\mathbb{Z}$ where p is prime. You must already be friends with the field \mathbb{Q} of rational numbers – fractions with integer numerator and nonzero integer denominator. And you know the field \mathbb{R} of real numbers from calculus: that is, limits of Cauchy sequences of rationals. We are not supposed to say the word “limit” as this is algebra. So we will not talk about constructing the field of real numbers \mathbb{R} . Ring theory topics include: definitions and basic properties of rings, fields, ideals, and functions between rings which preserve the ring operations (ring homomorphisms). We will also build new rings (quotient rings) whose elements are cosets $x + I$ of an ideal I in ring R , for x in ring R . Note that here R is an arbitrary ring, not necessarily the field of real numbers. We will look at rings of polynomials and their similarity to the ring of integers. We can do linear algebra for finite-dimensional vector spaces over arbitrary fields in a similar way to the linear algebra that is included in calculus sequences. Our favorite rings are the ring \mathbb{Z} of integers and the quotient ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n , in which x , is identified with all integers of the form $x + nk$, for integer k . Another favorite is the ring of Hamilton quaternions which is isomorphic to four-dimensional space over the real numbers with basis $1, i, j, k$ and with multiplication defined by $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$.

Historically, much of our subject came out of number theory and the desire to prove Fermat's last theorem by knowing about factorization into irreducibles in rings like $\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} \mid a, b \in \mathbb{Z}\}$, where m is a non-square integer. For example, it turns out that, when $m = -5$, we have two different factorizations:

$$2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5}).$$

So the fundamental theorem of arithmetic – true for \mathbb{Z} as is shown in Section 1.5 – is false for $\mathbb{Z}[\sqrt{-5}]$.

Assuming that such factorizations were unique, Lamé thought that he had proved Fermat's last theorem in 1847. Dedekind fixed up arithmetic in such rings by developing the arithmetic of ideals, which are certain sets of elements of the ring to be considered in Section 5.4. One then had (at least in rings of integers in algebraic number fields) unique factorization of ideals as products of prime ideals, up to order. Of course, Lamé's proof of Fermat's last theorem was still invalid (lame – sorry for that).

The favorite field of the average human mathematics student is the field of real numbers \mathbb{R} . A favorite finite field for a computer is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where $p = \text{prime}$. Of course you can define $\mathbb{Z}/n\mathbb{Z}$, for any positive integer n , but you only get a ring and not a field if n is not a prime. We consider $\mathbb{Z}/n\mathbb{Z}$ as a group under addition in Section 1.6. In Chapter 5 we view it as a ring with two operations, addition and multiplication.

Finite rings and fields were really invented by Gauss (1801) and earlier Euler (1750). Galois and Abel worked on field theory to figure out whether n th degree polynomial equations are solvable with equations involving only radicals $\sqrt[n]{a}$. In fact, finite fields are often called “Galois fields.”

The terminology of algebra was standardized by mathematicians such as Richard Dedekind and David Hilbert in the late 1800s. Much of abstract ring theory was developed in the 1920s by Emmy Noether. Discrimination against both women and Jews made it hard for her to publish. The work became well known thanks to B. L. Van der Waerden's two volumes [124] on modern algebra. Van der Waerden wrote these books after studying with Emmy Noether in 1924 in Göttingen. He had also heard lectures of Emil Artin in Hamburg earlier. See Edna E. Kramer [59] for more information on Noether and the other mathematicians who developed the view of algebra we are aiming to present.

The abstract theory of algebras (which are special sorts of rings) was applied to group representations by Emmy Noether in 1929. This has had a big impact on the way people do harmonic analysis, number theory, and physics. In particular, certain adelic group representations are central to the Wiles proof of Fermat's last theorem.

It would perhaps shock many pure mathematics students to learn how much algebra is part of the modern world of applied mathematics – both for good and ill. Google's motto: “Don't be evil,” has not always been the motto of those using algebra. Of course, the Google search engine itself is a triumph of modern linear algebra, as we shall see.

We will consider many applications of rings in Chapter 8. Section 8.1 concerns random number generators from finite rings and fields. These are used in simulations of natural phenomena. In prehistoric times like the 1950s sequences of random numbers came from tables like that published by the Rand corporation. Random numbers are intrinsic to Monte Carlo methods. These methods were named for a casino in Monaco by J. von Neumann and S. Ulam in the 1940s while working on the atomic bomb. Monte Carlo methods are useful in computational physics and chemistry (e.g., modeling the behavior of galaxies, weather on earth), engineering (e.g., simulating the impact of pollution), biology (simulating

the behavior of biological systems such as a cancer), statistics (hypothesis testing), game design, finance (e.g., to value options, analyze derivatives – the very thing that led to the horrible recession/depression of 2008), numerical mathematics (e.g., numerical integration, stochastic optimization), and the gerrymandering of voting districts.

In Section 8.2 we will show how the finite field with two elements and vector spaces over this field lead to error-correcting codes. These codes are used in DVDs and in the transmission of information between a Mars spacecraft and NASA on the earth. Section 8.3 concerns (among other things) the construction of Ramanujan graphs which can provide efficient communication networks.

Section 8.4 gives applications of the eigenvalues of matrices to Googling. Section 8.5 gives applications of elliptic curves over finite fields to cryptography.

The rush to abstraction of twentieth-century mathematics has had some odd consequences. One of the results of the abstract ring theory approach was to create such an abstract version of Fourier analysis that few can figure out what is going on. A similar thing has happened in number theory. On the other hand, modern algebra has often made it easier to see the forest for the trees by simplifying computations, removing subscripts, doing calculations once in the general case rather than many times, once for each example.

The height of abstraction was achieved in the algebra books of Nicolas Bourbaki (really a group of French mathematicians). I am using the Bourbaki notation for the fields of real numbers, complex numbers, rational numbers, and the ring of integers. But Bourbaki seems to have disliked pictures as well as applications. I do not remember seeing enough examples or history when I attempted to read Bourbaki's *Algebra* as an undergrad. In an interview, one of the members of Bourbaki, Pierre Cartier (see Marjorie Senechal [102]) said: "The Bourbaki were Puritans, and Puritans are strongly opposed to pictorial representations of truths of their faith." More information on Bourbaki as well as other fashions in mathematics can be found in Edna E. Kramer's history book [59]. She also includes a brief history of women in mathematics as well as the artificial separation between pure and applied mathematics.

As I said in my statement of goals, I will attempt to be as non-abstract as possible in this book and will seek to draw pictures in a subject where few pictures ever appear. I promise to give examples of every concept, but hope not to bury the reader in examples either, since I do aim for brevity. As I am a number theorist interested in matrix groups, there will be lots of numbers and matrices. Each chapter will have many exercises. It is important to do them – or as many of them as you can. Some exercises will be needed later in the book. The answers (mostly sketchy outlines) to odd-numbered exercises will be online hopefully. See my website. There may also be hints on others. No proof is intended to be very long. The computational problems might be slightly longer and sometimes impossible without the help of a computer. I will be using Mathematica, Scientific Workplace, and Group Explorer to help with computations.

Suggestions for Further Reading

A short list of possible references is: Garrett Birkhoff and Saunders MacLane [9], Larry L. Dornhoff and Franz E. Hohn [25], David S. Dummit and Richard M. Foote [28], Gertrude Ehrlich [29], John B. Fraleigh [32], Joseph A. Gallian [33], William J. Gilbert and W. Keith Nicholson [35], Israel N. Herstein [42], Audrey Terras [116]. There is also a free program: Group Explorer, which you can download and use to explore small groups. Another free but harder to use program is SAGE. I will be using Mathematica and Scientific Workplace. The Raspberry Pi computer (\$35) comes with Mathematica (and not much else). There are many

books on line as well. One example is Judson [50]. It includes computer exercises using SAGE. An on line group theory book making use of the Group Explorer program is that of Carter [12]. Wikipedia is often very useful – or just asking Google to answer a question. It is easier to be a student now than it was in my time – thanks to the multitude of resources to answer questions. On the other hand, it was nice just to have the one small book – in my case, Birkhoff and Maclane [9] – to deal with. And – perhaps needless to say – online sources can lie. Even the computer can lie – witness the arithmetic error in the Pentium chip that was revealed by number theorists' computations in the 1990s. But I have found that Wikipedia is usually very useful in its discussions of undergraduate mathematics, as is the mathematical software I have used.

It is often enlightening to look at more than one reference. Where something is mumbled about in one place, that same thing may be extremely clearly explained in another. Also feel free to read a book in a non-linear manner. If you are interested in a particular result or application, start there.

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