An Invitation to Model Theory

Model theory begins with an audacious idea: to consider statements about mathematical structures as mathematical objects of study in their own right. While inherently important as a branch of mathematical logic, it also enjoys connections to and applications in diverse branches of mathematics, including algebra, number theory and analysis. Despite this, traditional introductions to model theory assume a graduate-level background of the reader.

In this innovative textbook, Jonathan Kirby brings model theory to an undergraduate audience. The highlights of basic model theory are illustrated through examples from specific structures familiar from undergraduate mathematics, paying particular attention to definable sets throughout. With numerous exercises of varying difficulty, this is an accessible introduction to model theory and its place in mathematics.

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To Pirita, Lumia, Tapio, and Sakari

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Preface

This book is designed as an undergraduate or master's-level course in model theory. It has grown out of courses taught for many years at the University of Oxford and courses taught by me at UEA. The choice of material and presentation are based on pedagogical considerations, and I have tried to resist the temptation to be encyclopedic.

In this book, the main programme of model theory is to take a familiar mathematical structure and get an understanding of it in the following way. First, find an axiomatisation of its complete theory. Second, if possible, classify all the other models of the theory. Third, describe all the definable sets. As a result, model theory is presented as a set of tools for understanding structures, and the way the tools are applied to specific structures is as important as the tools themselves. This gives motivation to the subject and connects it to familiar material. Some readers may be more interested in the theory than in the applications, but my view is that even those who eventually wish to work in abstract model theory will get a better understanding of the basics by seeing them applied to examples.

Historically, model theory grew as a branch of mathematical logic, and the focus was mostly on logical issues, such as decidability. As model theory has found more applications and connections to other branches of mathematics, the study of definable sets has become more central.

I have tried to keep the book as self-contained as possible. Model theory requires a level of mathematical sophistication in terms of abstract, rigorous thinking, proofs, and algebraic thinking which students will normally have developed through previous courses in algebra, logic, or geometry, but there are few specific prerequisites. No topology is used, and almost no set theory is used. A brief chapter explains the basic cardinal arithmetic methods needed for some of the proofs, but with one or two exceptions, everything can be assumed to be countable. A familiarity with the use of basic algebraic ideas, such as bijections and homomorphisms or embeddings of groups, or rings, or vector spaces, is essential, but when algebraic examples such as rings and vector spaces are introduced, all the necessary definitions and facts are explained.

For the most part I have not given historical references for the material. There are good historical remarks at the end of each chapter in the books of Hodges [Hod93, Hod97] and Marker [Mar02], and I refer the reader to those.

Overview

The book is organised into six parts. Part I covers structures, languages, and automorphisms, introduces definable sets, and proves the essential preservation theorems.

Part II introduces theories and the programme of finding axiomatisations for structures. Context is given with axiomatisations of the complex and real fields and the natural numbers with addition and multiplication (Peano arithmetic). The compactness theorem is then introduced, and examples of its use are given. Part II concludes with the Henkin proof of the compactness theorem.

In Part III we show that even a complete theory can have many different models with the Löwenheim–Skolem theorems. The notion of categoricity is introduced via the example of vector spaces and is used to prove the completeness of an axiomatisation. Further applications are given to dense linear orders, where the back-and-forth method is introduced, and to the natural numbers with the successor function.

The idea of quantifier elimination is introduced in Part IV, and used to characterise the definable sets in dense linear orders and vector spaces. Boolean algebras are introduced via an investigation into the theory of power sets, partially ordered by the subset relation, and the definable subsets of a structure in any number n of variables is shown to be a Boolean algebra, called the Lindenbaum algebra of the theory. These Lindenbaum algebras are then worked out in the examples of vector spaces and for the real ordered field.

Parts V and VI go in different directions and do not depend on each other. Part V develops the notion of types, which is at the heart of modern model theory. The first goal is the Ryll–Nardzewski theorem, which characterises countably categorical theories as those for which there are only finitely many definable sets in any given number of variables. There is then a brief discussion of saturated models, leading to areas for further reading.

Part VI takes the model-theoretic techniques developed in the first four parts and applies it to the theory of algebraically closed fields. Two chapters explain the necessary algebraic background, and a third proves categoricity, completeness of the theory, and quantifier elimination. The next chapter explains how the definable sets correspond to algebraic varieties and constructible sets, and a final chapter gives a model theoretic proof of Hilbert's Nullstellensatz, which gives more information about the definable sets.

Suggestions for Using This Book as a Textbook

This book can be used for a course in many different ways. Figure 0.1, illustrating the dependencies between chapters, can be used as a guide to planning a course according to the lecturer's preferences.

Each chapter is intended to be of such a length that it can be taught in one hour (perhaps covering only the essentials) or in two hours (sometimes using material from the exercises). Everything depends on the first three chapters, so they should be covered first, unless the students have the prerequisite knowledge from predicate logic. The emphasis is on semantic ideas, including automorphisms, which is somewhat different from the usual emphasis in a first logic course, and I have successfully taught this material to a class of students of whom some had seen logic before and some not. For those who have not seen this material before, Chapter 3 in particular goes rather quickly. Chapters 4 and 5 carry on the study of formulas to give students more opportunity to consolidate their understanding.

Each chapter has several exercises at the end, which range from very easy consolidations of definitions to more substantial projects. Some exercises need more background than the other material in the chapter, and some exercises develop extension material.

A short model theory section of a mathematical logic course could have the compactness theorem and some applications as its goal and consist of Chapters 1, 2, 3, and 6, Section 8.1, and Chapter 9. This could be filled out with any of Chapters 4, 5, and 7, the rest of Chapter 8 or Chapter 11, or with applications from Chapters 14, 15, 16, or 19. The back-and-forth method is another highlight that could be reached quickly, by covering only the essential sections of Chapters 1, 3, 6, and 17.

The heart of this book is the study of definable sets. A course centred on those would consist of Parts I to IV, possibly omitting Chapters 5, 9, 11, and 16, and possibly adding Part VI.

A course aiming to cover the basics of types and the Ryll–Nardzewski theorem, without the emphasis on definable sets, could consist of Chapters 1, 2, 3, 6, 8, 10–15, and 23–25.

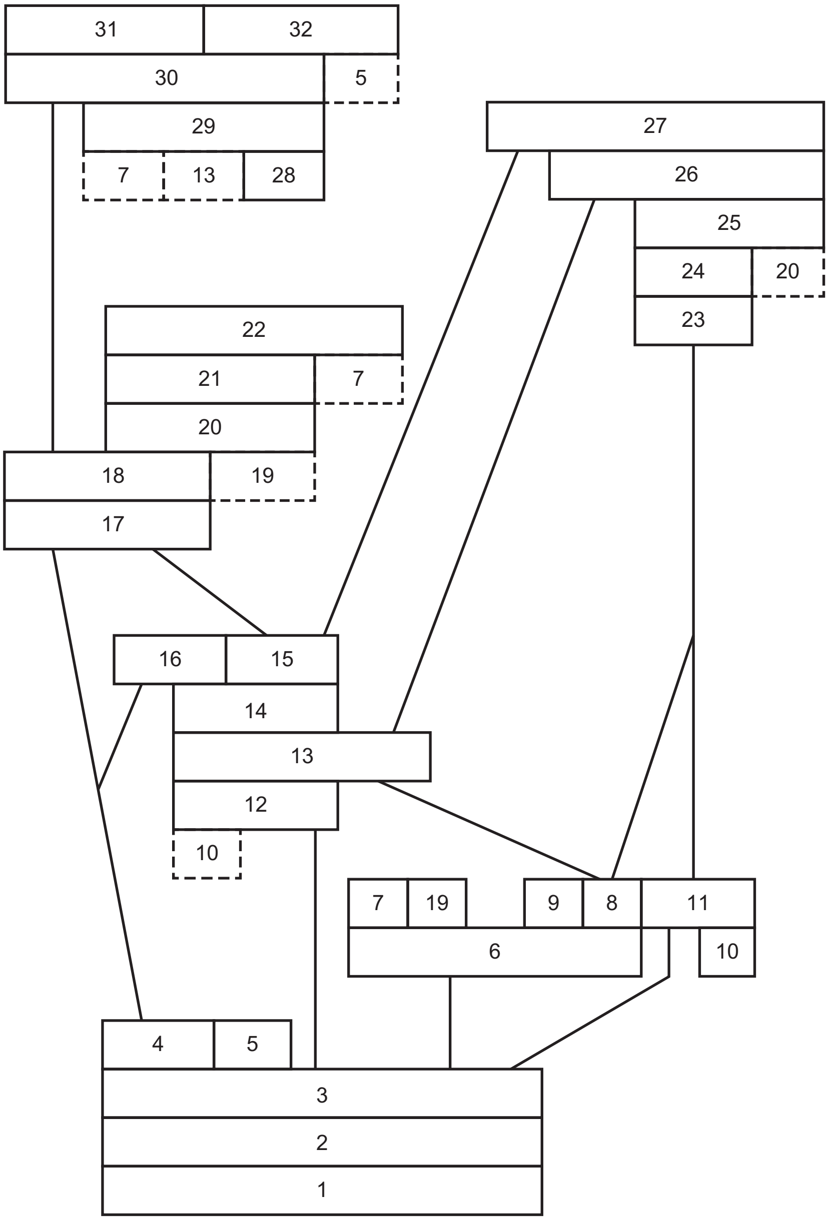

Figure 0.1 Chapter dependencies: In order to make the diagram planar, Chapters 5, 7, 10, 13, 19, and 20 have been drawn in two (or even three) places. As the diagram shows, Parts IV, V, and VI are largely independent of each other, with much of Part V also independent of Part III. While Chapter 25 uses the concept of Lindenbaum algebras from Chapter 20, that material does not rely on the rest of Parts III or IV.

For research students, Part VI, on algebraically closed fields, gives a pattern for carrying out the programme of understanding definable sets in a theory which is similar to the pattern which works in many other theories of fields, such as real-closed fields, differentially closed fields, separably closed fields, algebraically closed valued fields, exponentially closed fields, algebraically closed fields with an automorphism, and so on.

Further Reading

There are several more advanced textbooks in model theory, including those of Marker [Mar02], Hodges [Hod93, Hod97], Poizat [Poi00], Tent and Ziegler [TZ12], Sacks [Sac10], and Chang and Keisler [CK90]. Rothmaler [Rot00] is more at the level of this book, but there is more emphasis on algebra, especially groups. Väänänen [Vää11] introduces model theory via back-and-forth games, which complements the approach in this book. Bridge [Bri77] was based on the incarnation of the Oxford model theory course from the early 1970s, and the emphasis is much more on logic. Other suggestions for further reading are given through the book, particularly at the end of Parts IV, V, and VI.

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