Biomechanics

Thoroughly revised and updated for the second edition, this comprehensive textbook integrates basic and advanced concepts of mechanics with numerical methods and biomedical applications. Coverage is expanded to include a complete introduction to vector and tensor calculus, and new or fully updated chapters on biological materials and continuum mechanics, motion, deformation and rotation, and the constitutive modelling of solids and fluids. Topics such as kinematics, equilibrium, and stresses and strains are also included, as well as the mechanical behaviour of fibres and the analysis of one-dimensional continuous elastic media. Numerical solution procedures based on the finite element method are presented, with accompanying MATLAB-based software and dozens of new biomedical engineering examples and exercises allowing readers to practise and improve their skills. Solutions for instructors are also available online. This is the definitive guide for both undergraduate and graduate students taking courses in biomechanics.

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Cambridge Texts in Biomedical Engineering provide a forum for high-quality textbooks targeted at undergraduate and graduate courses in biomedical engineering. They cover a broad range of biomedical engineering topics from introductory texts to advanced topics, including biomechanics, physiology, biomedical instrumentation, imaging, signals and systems, cell engineering, and bioinformatics, as well as other relevant subjects, with a blending of theory and practice. While aiming primarily at biomedical engineering students, this series is also suitable for courses in broader disciplines in engineering, the life sciences and medicine.
“The increased number of exercises and examples used to bring the lectures alive and to illustrate the theory in biomedical applications make this second edition of the book ‘Biomechanics: Concepts and Computation’ definitely the reference to teach classical concepts of mechanics and computational modelling techniques for biomedical engineers at Bachelor level. The authors from Eindhoven University of Technology belong to one of the most prestigious Departments of Biomedical Engineering around the world, with a well-recognized expertise in Soft Tissue Biomechanics and Tissue Engineering. I have no hesitation in recommending that book that should be a prerequisite for any student studying biomechanics.”

Yohan Payan, Director of Research at Centre National de la Recherche Scientifique (CNRS), Université Grenoble Alpes

“A comprehensive textbook for learning all important concepts of biomechanics and their possible applications in sports and medicine. Students will enjoy the opportunity of learning computational modeling in biomechanics from scratch, needing only basic mathematical background. Instructors will appreciate the endless source of problems all resulting from successful experiences of teaching in the authors’ career. Definitely recommended in every library.”

Stéphane Avril, École des Mines, St Étienne

“‘Biomechanics: Concepts and Computation’ remains one of the strongest textbooks ever written in the field of biomechanical education. The theory in the book is thorough and rigorous, and is extremely well illustrated with numerous excellent exercises. I find the chapters describing numerical implementation and finite element formulations especially useful for translating the theory of tissue mechanics to bioengineering practice. I am using this book routinely in my undergraduate and graduate courses and will continue to do so with this second edition.”

Amit Gefen, Tel Aviv University
Biomechanics
Concepts and Computation
Second Edition

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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>About the Cover</td>
<td>xiii</td>
</tr>
<tr>
<td>Preface to the First Edition</td>
<td>xv</td>
</tr>
<tr>
<td>Preface to the Second Edition</td>
<td>xvii</td>
</tr>
<tr>
<td><strong>1 Vector and Tensor Calculus</strong></td>
<td></td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Definition of a Vector</td>
<td>1</td>
</tr>
<tr>
<td>1.3 Vector Operations</td>
<td>1</td>
</tr>
<tr>
<td>1.4 Decomposition of a Vector with Respect to a Basis</td>
<td>5</td>
</tr>
<tr>
<td>1.5 Some Mathematical Preliminaries on Second-Order Tensors</td>
<td>10</td>
</tr>
<tr>
<td>Exercises</td>
<td>13</td>
</tr>
<tr>
<td><strong>2 The Concepts of Force and Moment</strong></td>
<td></td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>16</td>
</tr>
<tr>
<td>2.2 Definition of a Force Vector</td>
<td>16</td>
</tr>
<tr>
<td>2.3 Newton’s Laws</td>
<td>18</td>
</tr>
<tr>
<td>2.4 Vector Operations on the Force Vector</td>
<td>19</td>
</tr>
<tr>
<td>2.5 Force Decomposition</td>
<td>20</td>
</tr>
<tr>
<td>2.6 Drawing Convention</td>
<td>24</td>
</tr>
<tr>
<td>2.7 The Concept of Moment</td>
<td>25</td>
</tr>
<tr>
<td>2.8 Definition of the Moment Vector</td>
<td>26</td>
</tr>
<tr>
<td>2.9 The Two-Dimensional Case</td>
<td>30</td>
</tr>
<tr>
<td>2.10 Drawing Convention for Moments in Three Dimensions</td>
<td>33</td>
</tr>
<tr>
<td>Exercises</td>
<td>34</td>
</tr>
<tr>
<td><strong>3 Static Equilibrium</strong></td>
<td></td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>39</td>
</tr>
<tr>
<td>3.2 Static Equilibrium Conditions</td>
<td>39</td>
</tr>
<tr>
<td>3.3 Free Body Diagram</td>
<td>42</td>
</tr>
<tr>
<td>Exercises</td>
<td>51</td>
</tr>
</tbody>
</table>
## Contents

### 4 The Mechanical Behaviour of Fibres

- 4.1 Introduction 56
- 4.2 Elastic Fibres in One Dimension 56
- 4.3 A Simple One-Dimensional Model of a Skeletal Muscle 59
- 4.4 Elastic Fibres in Three Dimensions 62
- 4.5 Small Fibre Stretches 69
- Exercises 73

### 5 Fibres: Time-Dependent Behaviour

- 5.1 Introduction 79
- 5.2 Viscous Behaviour 81
  - 5.2.1 Small Stretches: Linearization 84
- 5.3 Linear Visco-Elastic Behaviour 85
  - 5.3.1 Superposition and Proportionality 85
  - 5.3.2 Generalization for an Arbitrary Load History 88
  - 5.3.3 Visco-Elastic Models Based on Springs and Dashpots: Maxwell Model 92
  - 5.3.4 Visco-Elastic Models Based on Springs and Dashpots: Kelvin–Voigt Model 96
- 5.4 Harmonic Excitation of Visco-Elastic Materials 97
  - 5.4.1 The Storage and the Loss Modulus 97
  - 5.4.2 The Complex Modulus 99
  - 5.4.3 The Standard Linear Model 101
- 5.5 Appendix: Laplace and Fourier Transforms 106
- Exercises 108

### 6 Analysis of a One-Dimensional Continuous Elastic Medium

- 6.1 Introduction 116
- 6.2 Equilibrium in a Subsection of a Slender Structure 116
- 6.3 Stress and Strain 118
- 6.4 Elastic Stress–Strain Relation 121
- 6.5 Deformation of an Inhomogeneous Bar 122
- Exercises 129

### 7 Biological Materials and Continuum Mechanics

- 7.1 Introduction 133
- 7.2 Orientation in Space 134
- 7.3 Mass within the Volume $V$ 138
- 7.4 Scalar Fields 141
- 7.5 Vector Fields 144
Contents

7.6 Rigid Body Rotation 149
Exercises 151

8 Stress in Three-Dimensional Continuous Media 155
  8.1 Stress Vector 155
  8.2 From Stress to Force 156
  8.3 Equilibrium 157
  8.4 Stress Tensor 164
  8.5 Principal Stresses and Principal Stress Directions 172
  8.6 Mohr's Circles for the Stress State 175
  8.7 Hydrostatic Pressure and Deviatoric Stress 176
  8.8 Equivalent Stress 177
Exercises 178

9 Motion: Time as an Extra Dimension 183
  9.1 Introduction 183
  9.2 Geometrical Description of the Material Configuration 183
  9.3 Lagrangian and Eulerian Descriptions 185
  9.4 The Relation between the Material and Spatial Time Derivatives 188
  9.5 The Displacement Vector 190
  9.6 The Gradient Operator 192
  9.7 Extra Rigid Body Displacement 196
  9.8 Fluid Flow 198
Exercises 199

10 Deformation and Rotation, Deformation Rate and Spin 204
  10.1 Introduction 204
  10.2 A Material Line Segment in the Reference and Current Configurations 204
  10.3 The Stretch Ratio and Rotation 210
  10.4 Strain Measures and Strain Tensors and Matrices 214
  10.5 The Volume Change Factor 219
  10.6 Deformation Rate and Rotation Velocity 219
Exercises 222

11 Local Balance of Mass, Momentum and Energy 227
  11.1 Introduction 227
  11.2 The Local Balance of Mass 227
  11.3 The Local Balance of Momentum 228
### 11.4 The Local Balance of Mechanical Power 230

### 11.5 Lagrangian and Eulerian Descriptions of the Balance Equations 231

**Exercises** 233

### 12 Constitutive Modelling of Solids and Fluids 235

#### 12.1 Introduction 235

#### 12.2 Elastic Behaviour at Small Deformations and Rotations 236

#### 12.3 The Stored Internal Energy 242

#### 12.4 Elastic Behaviour at Large Deformations and/or Large Rotations 244

- **12.4.1 Material Frame Indifference** 244
- **12.4.2 Strain Energy Function** 250
- **12.4.3 The Incompressible Neo-Hookean Model** 252
- **12.4.4 The Incompressible Mooney–Rivlin Model** 255
- **12.4.5 Compressible Neo-Hookean Elastic Solid** 256

#### 12.5 Constitutive Modelling of Viscous Fluids 261

#### 12.6 Newtonian Fluids 262

#### 12.7 Non-Newtonian Fluids 263

#### 12.8 Diffusion and Filtration 264

**Exercises** 264

### 13 Solution Strategies for Solid and Fluid Mechanics Problems 270

#### 13.1 Introduction 270

#### 13.2 Solution Strategies for Deforming Solids 270

- **13.2.1 General Formulation for Solid Mechanics Problems** 271
- **13.2.2 Geometrical Linearity** 272
- **13.2.3 Linear Elasticity Theory, Dynamic** 273
- **13.2.4 Linear Elasticity Theory, Static** 273
- **13.2.5 Linear Plane Stress Theory, Static** 274
- **13.2.6 Boundary Conditions** 278

#### 13.3 Solution Strategies for Viscous Fluids 280

- **13.3.1 General Equations for Viscous Flow** 281
- **13.3.2 The Equations for a Newtonian Fluid** 282
- **13.3.3 Stationary Flow of an Incompressible Newtonian Fluid** 282
- **13.3.4 Boundary Conditions** 283
- **13.3.5 Elementary Analytical Solutions** 283

#### 13.4 Diffusion and Filtration 285

**Exercises** 287

### 14 Solution of the One-Dimensional Diffusion Equation by Means of the Finite Element Method 292

#### 14.1 Introduction 292
# Contents

14.2 The Diffusion Equation 293
14.3 Method of Weighted Residuals and Weak Form 295
14.4 Polynomial Interpolation 297
14.5 Galerkin Approximation 300
14.6 Solution of the Discrete Set of Equations 307
14.7 Isoparametric Elements and Numerical Integration 308
14.8 Basic Structure of a Finite Element Program 312
Exercises 319

15 Solution of the One-Dimensional Convection–Diffusion Equation by Means of the Finite Element Method 327
15.1 Introduction 327
15.2 The Convection–Diffusion Equation 327
15.3 Temporal Discretization 330
15.4 Spatial Discretization 333
Exercises 338

16 Solution of the Three-Dimensional Convection–Diffusion Equation by Means of the Finite Element Method 342
16.1 Introduction 342
16.2 Diffusion Equation 343
16.3 Divergence Theorem and Integration by Parts 344
16.4 Weak Form 345
16.5 Galerkin Discretization 345
16.6 Convection–Diffusion Equation 348
16.7 Isoparametric Elements and Numerical Integration 349
16.8 Example 353
Exercises 356

17 Shape Functions and Numerical Integration 363
17.1 Introduction 363
17.2 Isoparametric, Bi-Linear Quadrilateral Element 365
17.3 Linear Triangular Element 367
17.4 Lagrangian and Serendipity Elements
  17.4.1 Lagrangian Elements 371
  17.4.2 Serendipity Elements 373
17.5 Numerical Integration 373
Exercises 377
## Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>Infinitesimal Strain Elasticity Problems</td>
<td>382</td>
</tr>
<tr>
<td>18.1</td>
<td>Introduction</td>
<td>382</td>
</tr>
<tr>
<td>18.2</td>
<td>Linear Elasticity</td>
<td>382</td>
</tr>
<tr>
<td>18.3</td>
<td>Weak Formulation</td>
<td>384</td>
</tr>
<tr>
<td>18.4</td>
<td>Galerkin Discretization</td>
<td>385</td>
</tr>
<tr>
<td>18.5</td>
<td>Solution</td>
<td>391</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>394</td>
</tr>
<tr>
<td></td>
<td><strong>References</strong></td>
<td>399</td>
</tr>
<tr>
<td></td>
<td><strong>Index</strong></td>
<td>401</td>
</tr>
</tbody>
</table>
About the Cover

The cover contains images reflecting biomechanics research topics at the Eindhoven University of Technology. An important aspect of mechanics is experimental work to determine material properties and to validate models. The application field ranges from microscopic structures at the level of cells to larger organs like the heart. The core of biomechanics is constituted by models formulated in terms of partial differential equations and computer models to derive approximate solutions.

- **Main image**: Myogenic precursor cells have the ability to differentiate and fuse to form multinucleated myotubes. This differentiation process can be influenced by means of mechanical as well as biochemical stimuli. To monitor this process of early differentiation, immunohistochemical analyses are performed to provide information concerning morphology and localization of characteristic structural proteins of muscle cells. In the illustration, the sarcomeric proteins actin (red), and myosin (green) are shown. Nuclei are stained blue. Image courtesy of Mrs Marloes Langelaan.

- **Left top**: To study the effect of a mechanical load on the damage evolution of skeletal tissue, an in-vitro model system using tissue engineered muscle was developed. The image shows this muscle construct in a set-up on a confocal microscope. In the device the construct can be mechanically deformed by means of an indentor. Fluorescent identification of both necrotic and apoptotic cells can be established using different staining techniques. Image courtesy of Mrs Debby Gawlitta.

- **Left middle**: A three-dimensional finite element mesh of the human heart ventricles is shown. This mesh is used to solve the equations of motion for the beating heart. The model was used to study the effect of depolarization waves and mechanics in the paced heart. Image courtesy of Mr Roy Kerckhoffs.

- **Left bottom**: The equilibrium equations are derived from Newton’s laws and describe (quasi-)static force equilibrium in a three-dimensional continuum. See Eqs. (8.33), (8.34) and (8.35) in the present book.
Preface to the First Edition

In September 1997, an educational programme in Biomedical Engineering, unique in the Netherlands, started at the Eindhoven University of Technology, together with the University of Maastricht, as a logical step after almost two decades of research collaboration between both universities. This development culminated in the foundation of the Department of Biomedical Engineering in April 1999 and the creation of a graduate programme (MSc) in Biomedical Engineering in 2000 and Medical Engineering in 2002.

Already at the start of this educational programme, it was decided that a comprehensive course in biomechanics had to be part of the curriculum and that this course had to start right at the beginning of the Bachelor phase. A search for suitable material for this purpose showed that excellent biomechanics textbooks exist. But many of these books are very specialized to certain aspects of biomechanics. The more general textbooks address mechanical or civil engineers or physicists who wish to specialize in biomechanics, so these books include chapters or sections on biology and physiology. Almost all books that were found are at Masters or post-graduate level, requiring basic to sophisticated knowledge of mechanics and mathematics. At a more fundamental level, only books could be found that were written for mechanical and civil engineers.

We decided to write our own course material for the basic training in mechanics appropriate for our candidate biomedical engineers at Bachelor level, starting with the basic concepts of mechanics and ending with numerical solution procedures, based on the finite element method. The course material assembled in the current book comprises three courses for our biomedical engineering curriculum, distributed over the three years of their Bachelor studies. Chapters 1 to 6 mostly treat the basic concepts of forces, moments and equilibrium in a discrete context in the first year. Chapters 7 to 13 in the second year discuss the basis of continuum mechanics, and Chapters 14 to 18 in the third year are focussed on solving the field equations of mechanics using the finite element method.
What makes this book different from other basic mechanics or biomechanics treatises? Of course, as in standard books, there is the usual attention focussed on kinematics, equilibrium, stresses and strains. But several topics are discussed that are normally not found in one single textbook or only described briefly.

- Much attention is given to large deformations and rotations and non-linear constitutive equations (see Chapters 4, 9 and 10).
- A separate chapter is devoted to one-dimensional visco-elastic behaviour (Chapter 5).
- Special attention is given to long, slender, fibre-like structures (Chapter 4).
- The similarities and differences in describing the behaviour of solids and fluids and aspects of diffusion and filtration are discussed (Chapters 12 to 16).
- Basic concepts of mechanics and numerical solution strategies for partial differential equations are integrated in one single textbook (Chapters 14 to 18).

Because of the usually rather complex geometries (and non-linear aspects) found in biomechanical problems, hardly any relevant analytical solutions can be derived for the field equations, and approximate solutions have to be constructed. It is the opinion of the authors that, at Bachelor level, at least the basis for these numerical techniques has to be addressed.

In Chapters 14 to 18 extensive use is made of a finite element code written in MATLAB by one of the authors, which is especially developed as a tool for students. Applying this code requires that the user has a licence for the use of MATLAB, which can be obtained via MathWorks (www.mathworks.com). The finite element code, which is a set of MATLAB scripts, including manuals, is freely available and can be downloaded from the website: www.tue.nl/biomechanicsbook.
Preface to the Second Edition

Since 2009, when this book was published for the first time, we have been using it in our Biomechanics courses in the educational programme Biomedical Engineering, giving us hands-on experience with the book and the exercises. When we were investigating ideas for a second edition, we found that external reviewers were primarily asking for more examples and exercises, concurring with our own thoughts on the book. Over the years, we have assembled quite a number of examples that were often used to animate the lectures and to illustrate the theory in biomedical applications. At the same time, the number of available exercises increased considerably. Eventually, adding many of these examples and increasing the number of exercises are the most significant changes in the second edition.

The major changes in the text are:

- Mathematical preliminaries are now concentrated in Chapter 1 and no longer spread over different chapters.
- At some points in the original text, explanations were terse and too concise for students. Based on our experiences over the past eight years, we have extended the text at a number of points and, most importantly, added the earlier mentioned new examples. The biggest change is in Chapter 12, including a separate section on material frame indifference of constitutive equations and a more extensive treatment of hyperelastic materials.

The objectives of the book did not change. It is still meant to be a basic training in mechanics, appropriate for our candidate biomedical engineers at Bachelor level, starting with the basic concepts of mechanics and ending with numerical solution procedures. This book differs from most books on biomechanics, which are usually aimed at students with already considerable knowledge in continuum mechanics and wishing to enter the field of biomechanics. Consequently, those books pay great attention to biology and physiology. In contrast, we assume that students start at a very basic level in terms of mechanics, but already have substantial physiological and biological background knowledge.