

## Factorization Algebras in Quantum Field Theory

### Volume 1

Factorization algebras are local-to-global objects that play a role in classical and quantum field theory that is similar to the role of sheaves in geometry: they conveniently organize complicated information. Their local structure encompasses examples such as associative and vertex algebras; in these examples, their global structure encompasses Hochschild homology and conformal blocks.

In this first volume, the authors develop the theory of factorization algebras in depth, but with a focus upon examples exhibiting their use in field theory, such as the recovery of a vertex algebra from a chiral conformal field theory and a quantum group from Abelian Chern–Simons theory. Expositions of the relevant background in homological algebra, sheaves, and functional analysis are also included, thus making this book ideal for researchers and graduates working at the interface between mathematics and physics.

KEVIN COSTELLO is the Krembil Foundation William Rowan Hamilton Chair in Theoretical Physics at the Perimeter Institute in Waterloo, Ontario.

OWEN GWILLIAM is a postdoctoral fellow at the Max Planck Institute for Mathematics in Bonn.

## NEW MATHEMATICAL MONOGRAPHS

*Editorial Board*

Béla Bollobás, William Fulton, Frances Kirwan,  
 Peter Sarnak, Barry Simon, Burt Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit [www.cambridge.org/mathematics](http://www.cambridge.org/mathematics).

1. M. Cabanes and M. Enguehard *Representation Theory of Finite Reductive Groups*
2. J. B. Garnett and D. E. Marshall *Harmonic Measure*
3. P. Cohn *Free Ideal Rings and Localization in General Rings*
4. E. Bombieri and W. Gubler *Heights in Diophantine Geometry*
5. Y. J. Ionin and M. S. Shrikhande *Combinatorics of Symmetric Designs*
6. S. Berhanu, P. D. Cordaro and J. Hounie *An Introduction to Involutive Structures*
7. A. Shlapentokh *Hilbert's Tenth Problem*
8. G. Michler *Theory of Finite Simple Groups I*
9. A. Baker and G. Wüstholz *Logarithmic Forms and Diophantine Geometry*
10. P. Kronheimer and T. Mrowka *Monopoles and Three-Manifolds*
11. B. Bekka, P. de la Harpe and A. Valette *Kazhdan's Property (T)*
12. J. Neisendorfer *Algebraic Methods in Unstable Homotopy Theory*
13. M. Grandis *Directed Algebraic Topology*
14. G. Michler *Theory of Finite Simple Groups II*
15. R. Schertz *Complex Multiplication*
16. S. Bloch *Lectures on Algebraic Cycles (2nd Edition)*
17. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups*
18. T. Downarowicz *Entropy in Dynamical Systems*
19. C. Simpson *Homotopy Theory of Higher Categories*
20. E. Fricain and J. Mashreghi *The Theory of  $H(b)$  Spaces I*
21. E. Fricain and J. Mashreghi *The Theory of  $H(b)$  Spaces II*
22. J. Goubault-Larrecq *Non-Hausdorff Topology and Domain Theory*
23. J. Śniatycki *Differential Geometry of Singular Spaces and Reduction of Symmetry*
24. E. Riehl *Categorical Homotopy Theory*
25. B. A. Munson and I. Volić *Cubical Homotopy Theory*
26. B. Conrad, O. Gabber and G. Prasad *Pseudo-reductive Groups (2nd Edition)*
27. J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson *Sobolev Spaces on Metric Measure Spaces*
28. Y.-G. Oh *Symplectic Topology and Floer Homology I*
29. Y.-G. Oh *Symplectic Topology and Floer Homology II*
30. A. Bobrowski *Convergence of One-Parameter Operator Semigroups*
31. K. Costello and O. Gwilliam *Factorization Algebras in Quantum Field Theory I*
32. J.-H. Evertse and K. Györy *Discriminant Equations in Diophantine Number Theory*

Factorization Algebras in  
Quantum Field Theory  
Volume 1

KEVIN COSTELLO

*Perimeter Institute for Theoretical Physics, Waterloo, Ontario*

OWEN GWILLIAM

*Max Planck Institute for Mathematics, Bonn*



CAMBRIDGE  
UNIVERSITY PRESS

Cambridge University Press  
978-1-107-16310-2 — Factorization Algebras in Quantum Field Theory  
Kevin Costello, Owen Gwilliam  
Frontmatter  
[More Information](#)

**CAMBRIDGE**  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi - 110002, India  
79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9781107163102](http://www.cambridge.org/9781107163102)  
10.1017/9781316678626

© Kevin Costello and Owen Gwilliam 2017

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2017

*A catalogue record for this publication is available from the British Library*

ISBN 978-1-107-16310-2 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party Internet Web sites referred to in this publication and does not guarantee that any content on such Web sites is, or will remain, accurate or appropriate.

Cambridge University Press  
978-1-107-16310-2 — Factorization Algebras in Quantum Field Theory  
Kevin Costello, Owen Gwilliam  
Frontmatter  
[More Information](#)

---

TO LAUREN  
AND  
TO SOPHIE

Cambridge University Press

978-1-107-16310-2 — Factorization Algebras in Quantum Field Theory

Kevin Costello, Owen Gwilliam

Frontmatter

[More Information](#)

---

## Contents

---

<b>1</b>	<b>Introduction</b>	<i>page</i> 1
1.1	The Motivating Example of Quantum Mechanics	3
1.2	A Preliminary Definition of Prefactorization Algebras	8
1.3	Prefactorization Algebras in Quantum Field Theory	8
1.4	Comparisons with Other Formalizations of Quantum Field Theory	11
1.5	Overview of This Volume	16
1.6	Acknowledgments	18
	<b>PART I PREFACTORIZATION ALGEBRAS</b>	21
<b>2</b>	<b>From Gaussian Measures to Factorization Algebras</b>	23
2.1	Gaussian Integrals in Finite Dimensions	25
2.2	Divergence in Infinite Dimensions	27
2.3	The Prefactorization Structure on Observables	31
2.4	From Quantum to Classical	34
2.5	Correlation Functions	36
2.6	Further Results on Free Field Theories	39
2.7	Interacting Theories	40
<b>3</b>	<b>Prefactorization Algebras and Basic Examples</b>	44
3.1	Prefactorization Algebras	44
3.2	Associative Algebras from Prefactorization Algebras on $\mathbb{R}$	51
3.3	Modules as Defects	52
3.4	A Construction of the Universal Enveloping Algebra	59
3.5	Some Functional Analysis	62

3.6	The Factorization Envelope of a Sheaf of Lie Algebras	73
3.7	Equivariant Prefactorization Algebras	79
<b>PART II FIRST EXAMPLES OF FIELD THEORIES AND THEIR OBSERVABLES</b>		87
<b>4</b>	<b>Free Field Theories</b>	89
4.1	The Divergence Complex of a Measure	89
4.2	The Prefactorization Algebra of a Free Field Theory	93
4.3	Quantum Mechanics and the Weyl Algebra	106
4.4	Pushforward and Canonical Quantization	112
4.5	Abelian Chern–Simons Theory	115
4.6	Another Take on Quantizing Classical Observables	124
4.7	Correlation Functions	129
4.8	Translation-Invariant Prefactorization Algebras	131
4.9	States and Vacua for Translation Invariant Theories	139
<b>5</b>	<b>Holomorphic Field Theories and Vertex Algebras</b>	145
5.1	Vertex Algebras and Holomorphic Prefactorization Algebras on $\mathbb{C}$	145
5.2	Holomorphically Translation-Invariant Prefactorization Algebras	149
5.3	A General Method for Constructing Vertex Algebras	157
5.4	The $\beta\gamma$ System and Vertex Algebras	171
5.5	Kac–Moody Algebras and Factorization Envelopes	188
<b>PART III FACTORIZATION ALGEBRAS</b>		205
<b>6</b>	<b>Factorization Algebras: Definitions and Constructions</b>	207
6.1	Factorization Algebras	207
6.2	Factorization Algebras in Quantum Field Theory	215
6.3	Variant Definitions of Factorization Algebras	216
6.4	Locally Constant Factorization Algebras	220
6.5	Factorization Algebras from Cosheaves	225
6.6	Factorization Algebras from Local Lie Algebras	230
<b>7</b>	<b>Formal Aspects of Factorization Algebras</b>	232
7.1	Pushing Forward Factorization Algebras	232
7.2	Extension from a Basis	232



<i>Contents</i>	ix
7.3 Pulling Back Along an Open Immersion	240
7.4 Descent Along a Torsor	241
<b>8 Factorization Algebras: Examples</b>	243
8.1 Some Examples of Computations	243
8.2 Abelian Chern–Simons Theory and Quantum Groups	249
<i>Appendix A Background</i>	273
<i>Appendix B Functional Analysis</i>	310
<i>Appendix C Homological Algebra in Differentiable Vector Spaces</i>	351
<i>Appendix D The Atiyah–Bott Lemma</i>	374
<i>References</i>	377
<i>Index</i>	383