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INTRODUCTION TO BANACH SPACES: ANALYSIS AND PROBABILITY

This two-volume text provides a complete overview of the theory of Banach spaces, emphasising its interplay with classical and harmonic analysis (particularly Sidon sets) and probability. The authors give a full exposition of all results, as well as numerous exercises and comments to complement the text and aid graduate students in functional analysis. The book will also be an invaluable reference volume for researchers in analysis.

Volume 1 covers the basics of Banach space theory, operator theory in Banach spaces, harmonic analysis and probability. The authors also provide an annex devoted to compact Abelian groups.

Volume 2 focuses on applications of the tools presented in the first volume, including Dvoretzky's theorem, spaces without the approximation property, Gaussian processes and more. Four leading experts also provide surveys outlining major developments in the field since the publication of the original French edition.

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Introduction to Banach Spaces: Analysis and Probability

Volume 2

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Dedicated to the memory of
Jean-Pierre Kahane

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Daniel Li , Hervé Queffélec , Translated by Danièle Gibbons , Greg Gibbons
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Preface

This book is dedicated to the study of Banach spaces.

While this is an introduction, because we trace this study back to its origins, it is indeed a "specialized course",[1] in the sense that we assume that the reader is familiar with the general notions of Functional Analysis, as taught in late undergraduate or graduate university programs. Essentially, we assume that the reader is familiar with, for example, the first ten chapters of Rudin's book, Real and Complex Analysis (RUDIN 2); QUEFFÉLEC–ZUILY would also suffice.

It is also a "specialized course" because the subjects that we have chosen to study are treated in depth.

Moreover, as this is a textbook, we have taken the position to completely prove all the results "from scratch" (i.e. without referring within the proof to a "well-known result" or admitting a difficult auxiliary result), by including proofs of theorems in Analysis, often classical, that are not usually taught in French universities (as, for example, the interpolation theorems and the Marcel Riesz theorem in Chapter 7 of Volume 1, or Rademacher's theorem in Chapter 1 of Volume 2). The exceptions are a few results at the end of the chapters, which should be considered as complementary, and are not used in what follows.

We have also included a relatively lengthy first chapter introducing the fundamental notions of Probability.

As we have chosen to illustrate our subject with applications to "thin sets" coming from Harmonic Analysis, we have also included in Volume 1 an Annex devoted to compact Abelian groups.

This makes for quite a thick book,[2] but we hope that it can therefore be used without the reader having to constantly consult other texts.

[1] The French version of this book appeared in the collection "Cours Spécialisés" of the Société Mathématique de France.
[2] However, divided into two parts in the English version.

We have emphasized the aspects linked to Analysis and Probability; in particular, we have not addressed the geometric aspects at all; for these we refer, for example, to the classic Day, to Beauzamy or to more specialized books such as Benyamini–Lindenstrauss, Deville–Godefroy–Zizler or Pisier 2.

We have hardly touched on the study of operators on Banach spaces, for which we refer to Tomczak-Jaegermann and to Pisier 2; Diestel–Jarchow–Tonge and Pietsch–Wenzel are also texts in which the part devoted to operators is more important. Dunford–Schwartz remains a very good reference.

Even though Probability plays a large role here, this is not a text about Probability in Banach spaces, a subject perfectly covered in Ledoux–Talagrand.

Probability and Banach spaces were quick to get on well together. Although the study of random variables with values in Banach spaces began as early as the 1950s (R. Fortet and E. Mourier; we also cite Beck [1962]), their contribution to the study of Banach spaces themselves only appeared later, for example, citing only a few, Bretagnolle, Dacunha-Castelle and Krivine [1966], and Rosenthal [1970] and [1973]. However, it was only with the introduction of the notions of type and cotype of Banach spaces (Hoffmann-Jørgensen [1973], Maurey [1972 b] and [1972 c], Maurey and Pisier [1973]) that they proved to be intimately linked with Banach spaces.

Moreover, Probability also arises in Banach spaces by other aspects; notably it allows the derivation of the very important Dvoretzky's theorem (Chapter 1 of Volume 2), thanks to the concentration of measure phenomenon, a subject still highly topical (see the recent book of M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs **89**, AMS, 2001), dating back to Paul Lévy, and whose importance for Banach spaces was seen by Milman at the beginning of the 1970s.

We will also use Probability in a third manner, through the method of selectors, due to Erdös around 1955,[3] and afterwards used heavily by Bourgain, which allows us to make random constructions.

For all that, we do not limit ourselves to the probabilistic aspects; we also wish to show how the study of Banach spaces and of classical analysis interact (the construction by Davie, in Chapter 2 of Volume 2, of Banach spaces without the approximation property is typical in this regard); in particular we have concentrated on the application to thin sets in Harmonic Analysis.

Even if we have privileged these two points of view, we have nonetheless tried to give a global view of Banach spaces (with the exception of the

[3] Actually, this method traces back at least to Cramér [1935] and [1937].

geometric aspect, as already mentioned), with the concepts and fundamental results up through the end of the 1990s.

We point out that an interesting survey of what was known by the mid 1970s was given by Pełczyński and Bessaga [1979].

This book is divided into 14 chapters, preceded by a preliminary chapter and accompanied by an Annex. The first volume contains the first eight chapters, including the preliminary chapter and the Annex; the second volume contains the six remaining chapters. Moreover, it also contains three surveys, by G. Godefroy, O. Guédon and G. Pisier, on the major results and directions taken by Banach space theory since the publication of the French version of this book (2004), as well as an original paper of L. Rodríguez-Piazza on Sidon sets.

Each chapter is divided into sections, numbered by Roman numerals in capital letters (**I**, **II**, **III** etc.), and each section into subsections, numbered by Arabic numerals (**I.1** etc.). The theorems, propositions, corollaries, lemmas, definitions are numbered successively in the interior of each section; for example in Chapter 5 of Volume 1, Section IV they thus appear successively in the form: Proposition IV.1, Corollary IV.2, Definition IV.3, Theorem IV.4, Lemma IV.5, ignoring the subsections. If we need to refer to one chapter from another, the chapter containing the reference will be indicated.

At the end of each chapter, we have added comments. Certain of these cite complementary results; others provide a few indications of the origin of the theorems in the chapter. We have been told that "this is a good occasion to antagonize a good many colleagues, those not cited or incorrectly cited." We have done our best to correctly cite, in the proper chronological order, the authors of such and such result, of such and such proof. No doubt errors or omissions have been made; they are only due to the limits of our knowledge. When this is the case, we ask forgiveness in advance to the interested parties. We make no pretension to being exhaustive, nor to be working as historians. These indications should only be taken as incitements to the reader to refer back to the original articles and as complements to the contents of the course.

The chapters end with exercises. Many of these propose proofs of recent, and often important, results. In any case, we have attempted to decompose the proofs into a number of questions (which we hope are sufficient) so that the reader can complete all the details; just to make sure, in most cases we have indicated where to find the corresponding article or book.

The citations are presented in the following manner: if it concerns a book, the name of the author (or the authors) is given in small capitals, for example BANACH, followed by a number if there are several books by this author: RUDIN 3; if it concerns an article or contribution, it is cited by the name of the author or authors, followed in brackets by the year of publication, followed possibly by a lower-case letter: Salem and Zygmund [1954], James [1964 a].

We now come to a more precise description of what will be found in this book.

In the Preliminary Chapter, we quickly present some useful properties concerning the weak topology $w = \sigma(E, E^*)$ of a Banach space E and the weak* topology $w^* = \sigma(X^*, X)$ in a dual space X^*. Principally, we will prove the Eberlein–Šmulian theorem about weakly compact sets and the Krein–Milman theorem on extreme points. We then provide some information about filters and countable ordinals.

Chapter 1 of Volume 1 is intended for readers who have never been exposed to Probability Theory. With the exception of Section V concerning martingales, which will not be used until Chapter 7, its contents are quite elementary and very classical; let us say that they provide "Probability for Analysts." Moreover, in this book, we use little more than (but intensively!) Gaussian random variables (occasionally stable variables), and the Bernoulli or Rademacher random variables. The reader could refer to BARBE–LEDOUX or to REVUZ.

Section III provides the theorems of Kolmogorov for the convergence of series of independent random variables, and the equivalence theorem of Paul Lévy.

In Section IV, we show Khintchine's inequalities, which, even if elementary, are of capital importance for Analysis. We also find here the majorant theorem (Theorem IV.5) which will be very useful throughout the book.

Section V, a bit delicate for a novice reader of Probability, remains quite classical; we introduce martingales and prove Doob's theorems about their convergence.

In Chapter 2 (Volume 1) we begin the actual study of Banach spaces. We treat the Schauder bases, which provide a common and very practical tool.

After having shown in Section II that the projections associated with a basis are continuous and given a few examples (canonical bases of c_0, ℓ_p, Haar basis in $L^p(0,1)$, Schauder basis of $\mathcal{C}([0,1])$), we prove that the space $\mathcal{C}([0,1])$ is universal for the separable spaces, i.e. any separable Banach space is isometric to a subspace of $\mathcal{C}([0,1])$.

In Section III, we see how the use of bases, or more generally of basic sequences, allows us to obtain structural results; notably, thanks to the Bessaga–Pełczyński selection theorem, to show that any Banach space contains a subspace with a basis. We next show a few properties of the spaces c_0 and ℓ_p. Finally, we see how the spaces possessing a basis behave with respect to duality; this leads to the notions of shrinking bases and boundedly complete bases and to the corresponding structure theorems of James.

In Chapter 3 (Volume 1), we study the properties of unconditional convergence (i.e. commutative convergence) of series in Banach spaces.

After having given different characterizations of this convergence (Proposition II.2) and showed the Orlicz–Pettis theorem (Theorem II.3) in Section II, we introduce in Section III the notion of unconditional basis, and show, in particular, that the sequences of centered independent random variables are basic and unconditional in the spaces $L^p(\mathbb{P})$.

In Section IV, we study in particular the canonical basis of c_0, and prove the theorems of Bessaga and Pełczyński which, on one hand, characterize the presence of c_0 within a space by the existence of a scalarly summable sequence that is not summable, and, on the other hand, state that a dual space containing c_0 must contain ℓ_∞.

In Section V, we describe the James structure theorems characterizing, among the spaces having an unconditional basis, those containing c_0, or ℓ_1, or those that are reflexive.

All of the above work was done before 1960 and is now very classical.

In Section VI, we prove the Gowers dichotomy theorem, stating that every Banach space contains a subspace with an unconditional basis or a hereditarily indecomposable subspace (that is, none of its infinite-dimensional closed subspaces can be decomposed as a direct sum of infinite-dimensional closed subspaces). In addition, we provide a sketch of the proof of the homogeneous subspace theorem: every infinite-dimensional space that is isomorphic to all of its infinite-dimensional subspaces is isomorphic to ℓ_2.

In Chapter 4 (Volume 1), we study random variables with values in Banach spaces.

Section II essentially states that the properties of convergence in probability, almost surely, and in distribution, seen in Chapter 1 in the scalar case can be generalized “as such” for the vector-valued case. Prokhorov’s theorem (Theorem II.9) characterizes the families of relatively compact probabilities on a Polish space. The conditional expectation, more delicate to define than in the scalar case, is introduced, as well as martingales; the vectorial version of Doob’s theorem (Theorem II.12) then easily follows from the scalar case.

In Section III we describe the important symmetry principle, also known as the Paul Lévy maximal inequality, which allows us to obtain the equivalence theorem for series of independent Banach-valued random variables between convergence in distribution, almost sure and in probability.

The contraction principle of Section IV will be of fundamental importance for all that follows; in its quantitative version, it essentially states that for a real (respectively complex) Banach space E, the sequences of independent centered random variables in $L^p(E)$, $1 \leqslant p < +\infty$, are unconditional basic sequences with constant 2 (respectively 4).

In Section V, we generalize the scalar Khintchine inequalities to the vectorial case (Kahane inequalities); the proof is much more difficult than for the scalar case. These inequalities will turn out to be very important when we define the type and the cotype of Banach spaces (Chapter 5). The proof of the Kahane inequalities uses probabilistic arguments; in Subsection V.3, we will see how the use of the Walsh functions allowed Latała and Oleskiewicz, thanks to a hypercontractive property of certain operators (Proposition V.6), to obtain, in the case "$L^1 - L^2$," the best constant for these inequalities (Theorem V.4).

Chapter 5 (Volume 1) introduces the fundamental notions of type and cotype of Banach spaces.

It is now common practice to define these using Rademacher variables, but it is often more interesting to use Gaussian variables, notably for their invariance under rotation. We thus begin, in Section II, by providing some complements of Probability; we first define Gaussian vectors, and show their invariance under rotation (Proposition II.8); we take advantage of this to present the vectorial version of the central limit theorem, which we will use in Chapter 4 of Volume 2. We next prove the existence of p-stable variables, also to be used in Chapter 4 of Volume 2, and present the classical theorems of Schönberg on the kernels of positive type, and of Bochner, which characterizes the Fourier transforms of measures.

As notions of type and cotype are local, i.e. only involving the structure of finite dimensional subspaces, we give a few words in Section III to ultraproducts and to spaces finitely representable within another; we prove the local reflexivity theorem of Lindenstrauss and Rosenthal, stating, more or less, that the finite-dimensional subspaces of the bidual are almost isometric to subspaces of the space itself.

In Section IV, we define the type and cotype, give a few examples (type and cotype of L^p spaces, cotype 2 of the dual of a C^*-algebra), a few properties, and see how these notions behave with duality; this leads to the notion of K-convexity. We also show that in spaces having a non-trivial type, respectively cotype, we can, in the definition, replace the Rademacher variables by Gaussian variables (Theorem IV.8).

In Section V, we prove Kwapień's theorem, stating that a space is isomorphic to a Hilbert space if, and only if, it has at the same time type 2 and cotype 2; for this we first study the operators that factorize through a Hilbert space.

In Section VI, we present a few applications, and in particular show how to obtain the classical theorems of Paley and Carleman (Theorem VI.2).

In Chapter 6 (Volume 1), we will study a very important notion, that of a p-summing operator, brought out by Pietsch in 1967, and which soon afterward allowed Lindenstrauss and Pełczyński to highlight the importance of Grothendieck's theorem, which, even though proven in the mid 1950s, had not until then been properly understood.

We begin with an introduction showing that the 2-summing operators on a Hilbert space are the Hilbert–Schmidt operators.

In Section II, after having given the definition and pointed out the ideal property possessed by the space of p-summing operators, we prove the Pietsch factorization theorem, stating that the p-summing operators $T\colon X \to Y$ are those that factorize by the canonical injection (or rather its restriction to a subspace) of a space $\mathcal{C}(K)$ in $L^p(K, \mu)$, where K is a compact (Hausdorff) space and μ a regular probability measure on K; in particular the 2-summing operators factorize through a Hilbert space. It easily follows that the p-summing operators are weakly compact and are Dunford–Pettis operators. We next prove, thanks to Khintchine's inequalities, a theorem of Pietsch and Pełczyński stating that the Hilbert–Schmidt operators on a Hilbert space are not only 2-summing, but even 1-summing.

In Section III, we show Grothendieck's inequality (Theorem III.3), stating that scalar matrix inequalities are preserved when we replace the scalars by elements of a Hilbert space, losing at most a constant factor K_G, called the Grothendieck constant. We then prove Grothendieck's theorem: every operator of a space $L^1(\mu)$ into a Hilbert space is 1-summing. The proof is "local," meaning that it involves only the finite-dimensional subspaces; in passing we also show that the finite-dimensional subspaces of L^p spaces can be embedded, $(1 + \varepsilon)$-isomorphically, within spaces of sequences ℓ_p^N of finite dimension N. We then give the dual form of this theorem: every operator of a space $L^\infty(\nu)$ into a space $L^1(\mu)$ is 2-summing.

In Section IV, we present a number of results, originally proven in different ways, that can easily be obtained using the properties of p-summing operators (note that these do not depend on Grothendieck's theorem, contrary to what might be suggested by the order of the presentation): the Dvoretzky–Rogers theorem (every infinite-dimensional space contains at least one sequence unconditionally convergent but not absolutely convergent), John's theorem (the Banach–Mazur distance of every space of dimension n to the space ℓ_2^n is $\leqslant \sqrt{n}$), and the Kadeč–Snobar theorem (in any Banach space, there exists, on every subspace of dimension n, a projection of norm $\leqslant \sqrt{n}$). We then see that Grothendieck's theorem allows us to show that every normalized unconditional basis of ℓ_1 or of c_0 is equivalent to their canonical basis (this is also true for ℓ_2, but this case is easy).

Finally, Section V is devoted to Sidon sets (see Definition V.1). The fundamental example is that of Rademacher variables in the dual of the Cantor group $\Omega = \{-1, +1\}^{\mathbb{N}}$; another example is that of powers of 3 in $\mathbb{Z}$. We prove a certain number of properties, functional, arithmetical and combinatorial, demonstrating the "smallness" of Sidon sets; we show in passing the classical inequality of Bernstein. Grothendieck's theorem allows us to show that a set Λ is Sidon if and only if the space $\mathcal{C}_\Lambda$ is isomorphic to ℓ_1. We next present a theorem that is very important for the study of Sidon sets, Rider's theorem (Theorem V.18), which involves, instead of the uniform norm of polynomials, another norm $[\![\,.\,]\!]_R$, obtained by taking the expectation of random polynomials constructed by multiplying the coefficients by independent Rademacher variables. This allows us to obtain Drury's theorem (Theorem V.20), stating that the union of two Sidon sets is again a Sidon set, and the fact, due to Pisier, that Λ is a Sidon set as soon as $\mathcal{C}_\Lambda$ is of cotype 2; for this last result, we need to replace, in the norm $[\![\,.\,]\!]_R$, the Rademacher variables by Gaussian variables, and are led to show a property of integrability of Gaussian vectors, due to Fernique (Theorem V.26), a Gaussian version of the Khintchine–Kahane inequalities, which will also be useful in Chapter 6 of Volume 2.

In Chapter 7 (Volume 1), we present a few properties of the spaces L^p. In Section II, we study the space L^1. After having defined the notion of uniform integrability, we give a condition for a *sequence* of functions to be uniformly integrable (the Vitali–Hahn–Saks theorem), which allows us to deduce that the spaces $L^1(m)$ are weakly sequentially complete. We then characterize the weakly compact subsets of L^1 as being the weakly closed and uniformly integrable subsets (the Dunford–Pettis theorem). We conclude this section by showing that L^1 is not a subspace of a space with an unconditional basis. We will continue the study of L^1 in Chapter 4 of Volume 2; more specifically, we will examine the structure of its reflexive subspaces.

In Section III, we will see that the trigonometric system forms a basis of $L^p(0, 1)$ for $p > 1$. This is in fact an immediate consequence of the Marcel Riesz theorem, stating that the Riesz projection, or the Hilbert transform, is continuous on L^p for $p > 1$; most of Section III is hence devoted to the proof of this result. We have chosen not to prove it directly, but to reason by interpolation, allowing us to show in passing the Marcinkiewicz theorem, at the origin of real interpolation, as well as Kolmogorov's theorem stating that the Riesz projection is of weak type $(1, 1)$ (Theorem III.6). We conclude this section with a result of Orlicz (Corollary III.9) stating that the unconditional convergence of a series in L^p, for $1 \leqslant p \leqslant 2$, implies the convergence of the sum of the squares of the norms, implying that the trigonometric system is unconditional only for L^2.

In Section IV, we show, in contrast, that the Haar basis is unconditional in $L^p(0,1)$, for $1 < p < +\infty$. This unconditionality is linked to the facts that the Haar basis is a martingale difference and that martingale differences are unconditional in L^p, $1 < p < +\infty$ (Theorem IV.7). We also present some complements on martingales, notably on the behavior in L^p of the square function of a martingale (Theorem IV.6). The proof used here starts with the easy case, $p = 2$, and then passes successively, by doubling, to the cases $p = 4, 8, 16, \ldots$; we finish by interpolation, using the Riesz–Thorin theorem, previously shown in Subsection IV.1. To conclude this section, we study a particular property of the Haar basis, in a way rendering it extremal; for this, we need Lyapounov's theorem, stating that the image of vector measures with values in $\mathbb{R}^n$ is convex, and we prove this (Theorem IV.10).

Finally, the aim of Section V is to present another proof of Grothendieck's theorem, as a simple consequence of a theorem of Paley stating that $\sum_{k=1}^{+\infty} |\widehat{f}(2^k)|^2 < +\infty$ for every function $f \in H^1(\mathbb{T})$. For this, we very succinctly develop the theory of the spaces H^p, and prove the factorization theorem $H^1 = H^2 H^2$ (Theorem V.1) and the Frédéric and Marcel Riesz theorem. Grothendieck's theorem then follows from the fact that the operator $f \in A(\mathbb{T}) \mapsto \big(\widehat{f}(2^k)\big)_{k \geqslant 1} \in \ell_2$ is 1-summing and surjective (Theorem V.6).

In the Comments, we show that there is essentially only one space $L^1(m)$, if we assume it separable and the measure m atomless. We also give an alternative proof of the F. and M. Riesz theorem, due to Godefroy, using the notions of nicely placed sets and Shapiro sets.

Chapter 8 (Volume 1) is essentially devoted to Rosenthal's ℓ_1 theorem, discovered in 1974. It provides a way to very easily detect when a Banach space contains ℓ_1; it is a very general dichotomy theorem: in any Banach space, from every bounded sequence, we can extract either a weakly Cauchy subsequence or a subsequence equivalent to the canonical basis of ℓ_1. The majority of proofs currently given use a Ramsey-type theorem of infinite combinatorics, the Nash–Williams theorem; we proceed differently, by first showing, in Section II, by a method due to Debs in 1987, the Rosenthal–Bourgain–Fremlin–Talagrand theorem (Theorem II.3), which is also a dichotomy theorem for the extraction of subsequences, this time for the pointwise convergence of sequences of continuous functions on a Polish space. We then derive Rosenthal's theorem for real Banach spaces. The complex case does not follow immediately; Dor was the first to show how to adapt the proof of the real case to show the complex case; we use here a method due to Pajor [1983] which uses combinatorial arguments to obtain the complex case from the real case.

In Section III, we prove the Odell–Rosenthal theorem (Theorem III.2), stating that a separable Banach space X does not contain ℓ_1 if and only if every element of the unit ball $B_{X^{**}}$ of its bidual is the limit, for the weak* topology $\sigma(X^{**}, X^*)$, of a *sequence* of elements of the ball B_X of X. We next show a result of Pełczyński (Theorem III.5), by a method due to Dilworth, Girardi and Hagler [2000], stating that a Banach space contains ℓ_1 if and only if its dual contains $L^1(0,1)$, or if and only if this dual contains the space of measures $\mathcal{M}([0,1])$ on $[0,1]$.

The Annex (Volume 1) serves especially to give a general framework to the elements of Harmonic Analysis that we use in this book, even though we essentially use those of the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the Cantor group $\Omega = \{-1,+1\}^{\mathbb{N}^*}$ (sometimes its finite version), as well as those of finite Abelian groups in Chapter 2 of Volume 2. In Section II, we present various notions on Banach algebras: invertible elements, maximal ideals, spectrum of an element, spectral radius; characters of a commutative algebra; involutive Banach algebras and their positive linear functionals (Theorem II.12); C^*-algebras. We show that every commutative C^*-algebra is isometric to the algebra of continuous functions on a compact space (Theorem II.14).

Section III concerns compact Abelian groups G, which we assume metrizable for simplicity. We begin by proving the existence, and uniqueness, of the Haar measure, thanks to the use of a *strictly* convex and lower semi-continuous function on the set of probabilities on G equipped with the weak* topology (this approach requires the metrizability). We then give some results on convolution. We next define the dual group $\Gamma = \widehat{G}$ as the set of characters of G and note that the metrizability of G implies that the dual is countable; we then determine the dual of the Cantor group (Proposition III.9), and show that $\widehat{G}$ separates the points of G (Theorem III.10; in fact shown in Theorem III.16), and hence that the set $\mathcal{P}(G)$ of trigonometric polynomials, i.e. finite linear combinations of characters, is dense in $\mathcal{C}(G)$ and in $L^p(G)$ for $1 \leqslant p < +\infty$; moreover Γ is an orthonormal basis of $L^2(G)$. We next define the Fourier transform and show that it is injective. We conclude with results on approximate identities and on the Fejér and de la Vallée-Poussin kernels. We deduce that the norm of the convolution operator by a measure μ on $L^1(G)$, and also on $\mathcal{C}(G)$, is equal to the norm of μ.

The contents of Chapter 1 (Volume 2) are essentially of a local nature. We show a fundamental structure theorem concerning the finite-dimensional subspaces of Banach spaces, Dvoretzky's theorem, which states that every n-dimensional space E contains, for any $\varepsilon > 0$, "large" subspaces (of dimension on the order of $\log n$) which are $(1+\varepsilon)$-isomorphic to Hilbert spaces. The proof is based on an argument of compactness, the Dvoretzky–Rogers lemma, and, in an essential manner, on a probabilistic argument linked to the concentration of measure phenomenon.

We thus begin in Section II with some results from Probability; after reviewing the asymptotic behavior of Gaussian variables, we examine that of the associated maximal functions of independent Gaussian variables and their absolute value. We then prove the Maurey–Pisier deviation inequality (Theorem II.3), from which we can deduce their inequality of the concentration of measure (Theorem II.4). For this, we need Rademacher's theorem (more or less classical, but rarely taught) on the almost everywhere differentiability of Lipschitz functions in $\mathbb{R}^N$. This inequality of concentration of measure allows us to prove Dvoretzky's theorem in both real and complex spaces; nonetheless we also use another approach, due to Gordon, valid only for the real case, as it can easily be adapted to prove the isomorphic version of Milman and Schechtman (Subsection IV.5).

In Section III, we prove a theorem concerning the comparison of Gaussian vectors, in a form due to Maurey (Theorem III.3). This allows us to easily obtain some important probabilistic results: Slepian's lemma (Theorem III.5) and its variant, the Slepian–Sudakov lemma (Theorem III.4), to be used in the proof of Dvoretzky's theorem for the real case, and Sudakov's minoration (Theorem III.6); these three results will again serve, in an essential manner, in Chapters 3 and 6 (Volume 2). To prove Dvoretzky's theorem, we need to be able to compare stochastically not only the max of Gaussian variables, but also their minimax; this is the purpose of Gordon's theorem (Theorem III.7).

The actual proof of Dvoretzky's theorem is in Section IV. We in fact present two proofs; in both cases the principle is the same. First, we introduce the Gaussian dimension (Pisier calls it the concentration dimension) $d(X)$ of a Gaussian vector X (Definition IV.9). Dvoretzky's theorem is derived from what is known as the Gaussian version of Dvoretzky's theorem (Theorem IV.10), stating that when a Banach space E contains a Gaussian vector X made up of m independent Gaussian variables, then E contains, for any $\varepsilon > 0$, a subspace $(1+\varepsilon)$-Hilbertian of finite dimension controlled by the Gaussian dimension $d(X)$ of X. The derivation from the Gaussian version is based on the Dvoretzky–Rogers lemma (Proposition IV.1), itself based on a compactness property in the spaces of operators between finite-dimensional spaces, given by Lewis' lemma (Lemma IV.3). Next we prove Theorem IV.10. For this, we construct, out of independent copies of the Gaussian vector X, random operators on ℓ_2^k with values in E, where k is an appropriate multiple, dependent on ε, of $d(X)$. In the real case, the Slepian–Sudakov lemma allows us to limit from above the expectation of their norms, and Gordon's theorem to limit them from below. In the second proof (for the complex case, but for the real case as well), the two estimations are obtained at the same time by the

Maurey–Pisier concentration of measure inequality, by using the invariance of complex standard Gaussian vectors under the unitary group.

In the rest of Section IV, we examine certain examples; we see for example that the theorem is optimal for $E = \ell_\infty^n$. We also show that, with control of the cotype-2 constant of E, we can find, for any $\varepsilon > 0$, subspaces $(1+\varepsilon)$-Hilbertian of dimension proportional to that of E (Theorem IV.14). This will be useful in Chapter 6 (Volume 2). To conclude this section, we prove the isomorphic version (Theorem IV.15), due to Milman and Schechtman. This allows, in a real Banach space E of dimension n, to find, for any integer $k \leqslant n$, a subspace of dimension k, and whose distance to ℓ_2^k is this time no longer arbitrarily close to 1, but is instead controlled by an explicit function of n and k. For this, we *admit* a delicate result, due to Bourgain and Szarek, that is an improvement of the Dvoretzky–Rogers lemma, and then apply Gordon's theorem.

Finally, we show in Section V the Lindenstrauss–Tzafriri theorem, for whose proof Dvoretzky's theorem, associated with Kwapień's theorem (Chapter 5 of Volume 1, Section V), is essential; it states that if in a Banach space all the closed subspaces are complemented, then this space is isomorphic to a Hilbert space.

Chapter 2 (Volume 2), quite short, is dedicated to the construction by Davie of a separable Banach space without the approximation property. The problem of the existence of such a space was posed by Grothendieck in the mid 1950s; it generalized the old problem of the existence of a basis in every Banach space, which dates back to Banach himself, and was resolved in 1972 by Enflo. The construction given soon afterward by Davie is simpler than that of Enflo. It combines a probabilistic argument (method of selectors) with an argument from Harmonic Analysis concerning finite groups. It fits particularly well with the objectives of this book.

In Section II, we give a certain number of equivalent formulations of the approximation property, and Section III contains the actual construction. We show that, for any $p > 2$, ℓ_p contains a closed subspace without the approximation property. This is also the case for c_0 and for ℓ_p with $1 \leqslant p < 2$ (Szankowski), but the proof is more delicate; it can be found, for example, in Lindenstrauss–Tzafriri, Volume II, Theorem 1.g.4.

In Chapter 3 (Volume 2), we study in more detail Gaussian vectors, as well as the more general notion of Gaussian processes.

These are defined at the beginning of Section II. To each Gaussian process $X = (X_t)_{t \in T}$ we associate a (semi)-metric on T by setting $d_X(s,t) = \|X_s - X_t\|_2$; we show, with the aid of Slepian's lemma, that the condition $d_Y \leqslant d_X$ is sufficient to ensure that if X possesses a bounded version (respectively a continuous version), then so does Y (the Marcus–Shepp theorem).

In Section III, we define Brownian motion as an example of a Gaussian process.

Sections IV and V form the heart of this chapter.

In Section IV, we define the *entropy integral* associated with a Gaussian process: this is the integral, for $\varepsilon \in [0, +\infty[$, of $\sqrt{\log N(\varepsilon)}$, where $N(\varepsilon)$ is the entropy associated with the metric d_X of the process X, i.e. the minimum number of open d_X-balls necessary to cover T. The Dudley majoration theorem gives an upper bound for the expectation of the supremum of the modulus (absolute value) of a process with the aid of this entropy integral; a process has continuous paths as soon as the entropy integral is finite (Theorem IV.3). We then give an example showing that this condition is not necessary.

Next, in Section V, we see that, when the process is indexed by a compact metrizable Abelian group G and is *stationary*, i.e. its distribution does not change under translation, then the finiteness of the entropy integral $J(d)$ becomes necessary to have continuous trajectories, and $J(d)$ is, up to a constant, equivalent to $\mathbb{E}\big(\sup_{t\in G} |X_t|\big)$: this is the Fernique minoration theorem (Theorem V.4). We conclude the section by giving an equivalent form of the entropy integral (Proposition V.5) that will be useful in Chapter 6 (Volume 2).

Section VI returns to Banach spaces; we present the Elton–Pajor theorem (Theorem VI.12), which gives Elton's theorem: in a real Banach space, if there are N vectors $x_1, \ldots, x_N$ with norm $\leqslant 1$ such that the average of $\| \pm x_1 + \cdots \pm x_N\|$ over all choices of signs is $\geqslant \delta N$, then there is a subset of these vectors, of cardinality $N' \geqslant c(\delta) N$, which is equivalent to the canonical basis of $\ell_1^{N'}$, with constant $\beta(\delta)$ depending only on δ (Corollary VI.18). The proof uses probabilistic arguments: introduction of a Gaussian process and Dudley's majoration theorem, combinatorial arguments, notably Sauer's lemma (Proposition VI.3) and Chernov's inequality (Proposition VI.4), and volume arguments: Urysohn's inequality (Corollary VI.8), deduced from the Brunn–Minkowski inequality (Theorem VI.6), itself deduced from the Prékopa–Leindler inequality (Lemma VI.7). The complex version of Pajor requires several additional combinatorial lemmas (whose infinite-dimensional versions were used in Chapter 8 of Volume 1); it shows in particular that if a complex Banach space contains δ-isomorphically, as a real Banach space, the space ℓ_1^N, then it contains, in the complex sense, the complex space ℓ_1^{cN}, where c depends only on δ (Corollary VI.21).

In Chapter 4 (Volume 2), we concentrate on the reflexive subspaces of L^1.

In Section II, we first see that the reflexive subspaces of L^1 are those for which the topology of the norm coincides with that of the convergence in measure (the Kadeč–Pełczyński theorem) and that, in consequence, any non-reflexive subspace contains a *complemented* subspace isomorphic to ℓ_1 (Corollary II.6).

We then examine their local structure. Even though *a priori*, as L^1 is weakly sequentially complete (Chapter 7 of Volume 1, Theorem II.6), its reflexive subspaces are those that do not contain ℓ_1, by the Rosenthal ℓ_1 theorem (Chapter 8 of Volume 1), in fact we have much more: the reflexive subspaces of L^1 are those not containing ℓ_1^n's uniformly (Theorem II.7). We then show that the Banach spaces that do not contain ℓ_1^n's uniformly are exactly those with a type $p > 1$ (Theorem II.8, of Pisier), so that the reflexive subspaces of L^1 have a non-trivial type $p > 1$ (Corollary II.9).

In Section III, we present some examples of reflexive subspaces. We first see that, for $1 < p \leqslant 2$, the sequences of independent p-stable variables generate isometrically ℓ_p in the real L^1 space (Theorem III.1). We then succinctly study the $\Lambda(q)$-sets, which are the reflexive and translation-invariant subspaces of $L^1(\mathbb{T})$. In particular we prove the Rudin transfer theorem, stating that the properties of Rademacher functions in the dual of the Cantor group are transferred to all the Sidon sets (Theorem III.10), so that, thanks to the Khintchine inequalities, every Sidon set Λ is a $\Lambda(q)$-set for any $q < +\infty$, and, more precisely, $\|f\|_q \leqslant C\,S(\Lambda)\sqrt{q}\,\|f\|_2$ for every trigonometric polynomial f with spectrum in Λ, where C is a numerical constant and $S(\Lambda)$ is the Sidon constant of Λ (Theorem III.11). The converse, due to Pisier, is shown in two different ways, first, in Chapter 5 (Volume 2), with a method of random extraction due to Bourgain, and then, in Chapter 6 (Volume 2), with the aid of Gaussian processes, which was the original proof of Pisier.

Section IV is devoted to the deep theorem of Rosenthal showing that the reflexive subspaces of L^1 embed in L^p, for some $p > 1$ (Theorem IV.1). We use in the proof the Maurey factorization theorem (Theorem IV.2), that Maurey isolated from the original proof of Rosenthal. We thus deduce that every $\Lambda(1)$-set is in fact $\Lambda(q)$ for some $q > 1$ (Corollary IV.3).

In Section V, we study the finite-dimensional subspaces of L^1, and, more precisely, the dimension n of spaces ℓ_1^n that they can contain (Theorem V.2, of Talagrand). To make this statement more precise, we first need to study the K-convexity constant of finite-dimensional spaces (Theorem V.3), and in particular of those of L^1 (Theorem V.5). We also see that, up to a constant, nothing changes in the definition if the Rademacher variables are replaced by Gaussian variables (Theorem V.8). We need to prove an auxiliary result, due to Lewis (Theorem V.9). The proof of Talagrand's theorem is then based on the method of selectors, as well as Pajor's theorem from the preceding chapter to reduce to the real case.

Chapter 5 (Volume 2) contains three results of Bourgain illustrating the method of selectors.

This method was already used, in Chapters 2 and Chapter 4 of Volume 2; it involves selecting an independent sequence of Bernoulli variables $\varepsilon_1, \ldots, \varepsilon_n$, taking on the values 0 and 1 with a certain probability, and then making constructions by randomly choosing the set $\Lambda(\omega)$ of integers $k \leqslant n$ for which $\varepsilon_k(\omega)$ takes the value 1.

Section II treats the extraction of *quasi-independent sets*; these are particular Sidon sets, defined in an arithmetical manner, and whose Sidon constant is bounded by a fixed constant ($\leqslant 8$). We prove a theorem of Pisier stating that a set Λ is Sidon if, and only if, there exists a constant δ such that every finite subset A of Λ, not reduced to $\{0\}$, contains a quasi-independent subset B of cardinality $|B| \geqslant \delta\,|A|$ (Theorem II.3). In fact, we show that from every finite subset A, not reduced to $\{0\}$, we can extract a quasi-independent subset B of cardinality $|B| \geqslant K\,(|A|/\psi_A)^2$, where K is a numerical constant and ψ_A depends only on A (Theorem II.6). As an immediate consequence we have Drury's theorem (Corollary II.4), and we easily obtain Pisier's theorem (Theorem II.13), the converse of Rudin's theorem seen in Chapter 4 of Volume 2, as well as Rider's theorem (Theorem II.14).

In Section III, we show that, for any $N \geqslant 1$, there exists a subset $\Lambda \subseteq \mathbb{N}^*$ of cardinality N such that $\big\| \sum_{k\in\Lambda} \sin kx \big\|_\infty \leqslant C_0\, N^{2/3}$, where C_0 is a numerical constant (Theorem III.1). The interest in this result is linked to the vector-valued Hilbert transform: if E is a Banach space of finite dimension N, John's theorem immediately implies that the Hilbert transform with values in E has a norm $\leqslant \sqrt{N}$ in $\mathcal{L}\big(L^2(E)\big)$; if $E = \ell_1^N$, this norm is dominated by $\log N$; the preceding result shows that for every $N \geqslant 1$, we can find a Banach space E of dimension N so that this norm dominates $N^{1/3}$.

In Section IV, we show that the majoration $K(X) \leqslant C \log n$ for the K-convexity constant of spaces of dimension n seen in Chapter 4 (Volume 2) can essentially not be improved (Theorem IV.1).

Chapter 6 (Volume 2) is for the most part devoted to Pisier's space $\mathcal{C}^{as}$.

In Section II, we prove two results that will be needed in the next section. The first is the Itô–Nisio theorem, stating that, when $\sum_{n\geqslant 1} X_n$ is a series of independent symmetric random variables with values in $\mathcal{C}(K)$, where K is a metrizable compact space, such that, for every $t \in K$, the series $\sum_{n=1}^{+\infty} X_n(\cdot, t)$ converges almost surely to X_t, and in addition we assume that the process $(X_t)_{t\in K}$ has a continuous version, then the series is almost surely uniformly convergent (Theorem II.2). We then show a Tauberian theorem (the Marcinkiewicz–Zygmund–Kahane theorem): if $\sum_{n\geqslant 1} X_n$ is a series of independent symmetric random variables with values in a Banach space E, then the fact that it is almost surely bounded (respectively almost surely convergent) according to a summation procedure implies that this holds in the usual sense (Theorem II.4).

In Section III, the space $\mathcal{C}^{as}$ is defined: let G be a compact metrizable Abelian group and $\Gamma = \{\gamma_n\,;\ n \geqslant 1\}$ its dual group; let $(Z_n)_{n\geqslant 1}$ be a standard sequence of independent complex Gaussian variables; then $\mathcal{C}^{as}(G)$ is the space of all the functions $f \in L^2(G)$ for which, almost surely in ω, the sum of the series $\sum_{n\geqslant 1} Z_n(\omega)\widehat{f}(\gamma_n)\gamma_n$ is a continuous function $f^\omega \in \mathcal{C}(G)$. Theorem III.1 gives several equivalent formulations (one of these being Billard's theorem). Equipped with the norm defined by $[\![f]\!] = \sup_{N\geqslant 1} \mathbb{E}\big\| \sum_{n=1}^{N} Z_n\widehat{f}(\gamma_n)\gamma_n \big\|_\infty$, which is $\geqslant \|f\|_2$, $\mathcal{C}^{as}(G)$ is a Banach space for which the characters $\gamma_n \in \Gamma$ form a 1-unconditional basis (Theorem III.4). The Marcus–Pisier theorem (Theorem III.5) allows the Gaussian variables Z_n in the definition to be replaced by Rademacher variables; the proof uses the Dudley majoration theorem and the Fernique minoration theorem. The fundamental result concerning $\mathcal{C}^{as}$ is Theorem III.9. It establishes a duality between $\mathcal{C}^{as}$ and the space of multipliers M_{2,Ψ_2} from $L^2(G)$ to $L^{\Psi_2}(G)$, where Ψ_2 is the Orlicz function $\Psi_2(x) = \mathrm{e}^{x^2} - 1$, and shows that with this duality M_{2,Ψ_2} can be identified, isomorphically, with the dual of $\mathcal{C}^{as}$. The first part of the theorem again uses the Fernique minoration theorem; the second part is more delicate, and in addition to the Marcus–Pisier theorem, requires several auxiliary results. Thanks to this duality, we easily establish a result of Salem and Zygmund that gives upper and lower bounds of the norm $[\![\,.\,]\!]$ of a sum of exponentials (Proposition III.13).

In Section IV we present two more applications of $\mathcal{C}^{as}$. First we prove a theorem due to Pisier, a converse to Rudin's theorem (Chapter 4 of Volume 2), that characterizes Sidon sets Λ as those for which $\|f\|_q \leqslant C\sqrt{q}\,\|f\|_2$ for every trigonometric polynomial f with spectrum in Λ (Theorem IV.1); note that this uses only the existence of a duality between $\mathcal{C}^{as}$ and M_{2,Ψ_2}, and not the fact that M_{2,Ψ_2} is the dual of $\mathcal{C}^{as}$, and the Gaussian Rider theorem seen in Chapter 6 of Volume 1. Next, this space provides a response to the Katznelson dichotomy problem. Katznelson showed that only the real-analytic functions operate on the Wiener algebra $A(\mathbb{T})$, while it is clear that all continuous functions operate on $\mathcal{C}(\mathbb{T})$; the problem was to know if, for every Banach algebra B possessing certain "nice" properties, and such that $A(\mathbb{T}) \subseteq B \subseteq \mathcal{C}(\mathbb{T})$, either all continuous functions operate on B or only the analytic functions operate on B. Zafran found a counterexample to this conjecture; Theorem IV.2 (Pisier) reinforces the result of Zafran: $\mathfrak{P} = \mathcal{C}^{as}(\mathbb{T}) \cap \mathcal{C}(\mathbb{T})$, equipped with the norm $\|f\|_{\mathfrak{P}} = 8\,\|f\|_\infty + [\![f]\!]$, is a Banach algebra possessing the required qualities, but in which all the Lipschitz functions operate.

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To conclude Chapter 5, we prove the Bourgain–Milman theorem (Theorem V.1): Λ is a Sidon set as soon as $\mathcal{C}_\Lambda$ has a finite cotype (we have already seen in Chapter 6, Volume 1, that this is the case if the cotype is 2). The proof uses the notions of *Banach diameter* $n(E)$ of a finite-dimensional Banach space E (Definition V.2) and *arithmetic diameter* for the finite subsets of the dual of a compact metrizable Abelian group G, where the latter is the entropy number $\overline{N_A}(1/2)$ for a pseudo-metric $\overline{d_A}$ on G, associated with the finite subset A of the dual of G (Definition V.3). Using Dvoretzky's theorem for cotype-2 spaces (in fact for ℓ_1), the proof combines Theorem V.4 (of Maurey), which gives a lower bound for $n(E)$ as a fonction of the cotype constant of E, and Theorem V.5 (of Pisier): if Λ is a finite subset in the dual of G and if $\overline{N_A}(\delta) \geqslant \mathrm{e}^{\delta\,|A|}$ for every $A \subseteq \Lambda$, then the Sidon constant of Λ is bounded above by $a\,\delta^{-b}$, where $a, b > 0$ are numerical constants.

In the Comments, Section VI, as an application of random Fourier series, we prove two more results: one concerning functions of the Nevanlinna class (Theorem VI.1), and the other about random Dirichlet series (Theorem VI.2).

For the reader who would like to dig a bit deeper, we refer to the works cited in the bibliography, and in particular to the recent Handbook of the Geometry of Banach Spaces, Vols. I and II.

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Conventions

(1) In this book, the set $\mathbb{N}$ of natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* = \{1, 2, \ldots\}$.
(2) Compact spaces are always assumed to be Hausdorff.