CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

A COURSE IN FINITE GROUP REPRESENTATION THEORY

This graduate-level text provides a thorough grounding in the representation theory of finite groups over fields and rings. The book provides a balanced and comprehensive account of the subject, detailing the methods needed to analyze representations that arise in many areas of mathematics. Key topics include the construction and use of character tables, the role of induction and restriction, projective and simple modules for group algebras, indecomposable representations, Brauer characters, and block theory.

This classroom-tested text provides motivation through a large number of worked examples, with exercises at the end of each chapter that test the reader's knowledge, provide further examples and practice, and include results not proven in the text. Prerequisites include a graduate course in abstract algebra and familiarity with the properties of groups, rings, field extensions, and linear algebra.

Peter Webb is a Professor of Mathematics at the University of Minnesota. His research interests focus on the interactions between group theory and other areas of algebra, combinatorics, and topology.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.
For a complete series listing, visit: www.cambridge.org/mathematics.

Already published

124 K. Lux & H. Pahlings *Representations of groups*
125 K. S. Kedlaya *p-adic differential equations*
126 R. Beals & R. Wong *Special functions*
127 E. de Faria & W. de Melo *Mathematical aspects of quantum field theory*
128 A. Terras *Zeta functions of graphs*
129 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, I*
130 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, II*
131 D. A. Craven *The theory of fusion systems*
132 J. Väänänen *Models and games*
133 G. Malle & D. Testerman *Linear algebraic groups and finite groups of Lie type*
134 P. Li *Geometric analysis*
135 F. Maggi *Sets of finite perimeter and geometric variational problems*
136 M. Brodmann & R. Y. Sharp *Local cohomology (2nd Edition)*
137 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, I*
138 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, II*
139 B. Helffer *Spectral theory and its applications*
140 R. Pemantle & M. C. Wilson *Analytic combinatorics in several variables*
141 B. Branner & N. Fagella *Quasiconformal surgery in holomorphic dynamics*
142 R. M. Dudley *Uniform central limit theorems (2nd Edition)*
143 T. Leinster *Basic category theory*
144 I. Arzhantsev, U. Derenthal, J. Hausen & A. Laface *Cox rings*
145 M. Viana *Lectures on Lyapunov exponents*
146 J.-H. Evertse & K. Győry *Unit equations in Diophantine number theory*
147 A. Prasad *Representation theory*
148 S. R. Garcia, J. Mashreghi & W. T. Ross *Introduction to model spaces and their operators*
149 C. Godsil & K. Meagher *Erdős–Ko–Rado theorems: Algebraic approaches*
150 P. Mattila *Fourier analysis and Hausdorff dimension*
151 M. Viana & K. Oliveira *Foundations of ergodic theory*
152 V. I. Paulsen & M. Raghupathi *An introduction to the theory of reproducing kernel Hilbert spaces*
153 R. Beals & R. Wong *Special functions and orthogonal polynomials*
154 V. Jurdjevic *Optimal control and geometry: Integrable systems*
155 G. Pisier *Martingales in Banach spaces*
156 C. T. C. Wall *Differential topology*
157 J. C. Robinson, J. L. Rodrigo & W. Sadowski *The three-dimensional Navier–Stokes equations*
158 D. Huybrechts *Lectures on K3 surfaces*
159 H. Matsumoto & S. Taniguchi *Stochastic analysis*
160 A. Borodin & G. Olshanski *Representations of the infinite symmetric group*
161 P. Webb *A course in finite group representation theory*
162 C. J. Bishop & Y. Peres *Fractals in probability and analysis*
163 A. Bovier *Gaussian processes on trees*

A Course in Finite Group Representation Theory

PETER WEBB
University of Minnesota

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107162396

First published 2016

Printed in the United States of America by Sheridan Books, Inc.

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Webb, Peter, 1954–
Title: A course in finite group representation theory / Peter Webb, University of Minnesota.
Description: Cambridge : Cambridge University Press, [2016] | Series: Cambridge studies in advanced mathematics | Includes bibliographical references and index.
Identifiers: LCCN 2016013913 | ISBN 9781107162396 (hardback : alk. paper)
Subjects: LCSH: Finite groups – Textbooks. | Group theory – Textbooks. | Modules (Algebra) – Textbooks. | Representations of groups – Textbooks. | Rings (Algebra) – Textbooks.
Classification: LCC QA177 .W43 2016 | DDC 512/.23–dc23
LC record available at http://lccn.loc.gov/2016013913

ISBN 978-1-107-16239-6 Hardback

Contents

Preface *page* ix

1 Representations, Maschke's Theorem, and Semisimplicity 1
1.1 Definitions and Examples 1
1.2 Semisimple Representations 7
1.3 Summary of Chapter 1 11
1.4 Exercises for Chapter 1 11

2 The Structure of Algebras for Which Every Module Is Semisimple 15
2.1 Schur's Lemma and Wedderburn's Theorem 15
2.2 Summary of Chapter 2 20
2.3 Exercises for Chapter 2 20

3 Characters 24
3.1 The Character Table 24
3.2 Orthogonality Relations and Bilinear Forms 30
3.3 Consequences of the Orthogonality Relations 33
3.4 The Number of Simple Characters 38
3.5 Algebraic Integers and Divisibility of Character Degrees 42
3.6 The Matrix Summands of the Complex Group Algebra 45
3.7 Burnside's $p^a q^b$ Theorem 48
3.8 Summary of Chapter 3 50
3.9 Exercises for Chapter 3 51

4 The Construction of Modules and Characters 56
4.1 Cyclic Groups and Direct Products 56

4.2 Lifting (or Inflating) from a Quotient Group 59
4.3 Induction and Restriction 60
4.4 Symmetric and Exterior Powers 70
4.5 The Construction of Character Tables 75
4.6 Summary of Chapter 4 75
4.7 Exercises for Chapter 4 76

5 More on Induction and Restriction: Theorems of Mackey and Clifford 82
5.1 Double Cosets 82
5.2 Mackey's Theorem 84
5.3 Clifford's Theorem 87
5.4 Summary of Chapter 5 90
5.5 Exercises for Chapter 5 91

6 Representations of *p*-Groups in Characteristic *p* and the Radical 95
6.1 Cyclic *p*-Groups 95
6.2 Simple Modules for Groups with Normal *p*-Subgroups 98
6.3 Radicals, Socles, and the Augmentation Ideal 100
6.4 Jennings's Theorem 106
6.5 Summary of Chapter 6 108
6.6 Exercises for Chapter 6 108

7 Projective Modules for Finite-Dimensional Algebras 115
7.1 Characterizations of Projective and Injective Modules 115
7.2 Projectives by Means of Idempotents 119
7.3 Projective Covers, Nakayama's Lemma, and Lifting of Idempotents 122
7.4 The Cartan Matrix 130
7.5 Summary of Chapter 7 132
7.6 Exercises for Chapter 7 132

8 Projective Modules for Group Algebras 135
8.1 The Behavior of Projective Modules under Induction, Restriction, and Tensor Product 135
8.2 Projective and Simple Modules for Direct Products of a *p*-Group and a p'-Group 138
8.3 Projective Modules for Groups with a Normal Sylow *p*-Subgroup 140
8.4 Projective Modules for Groups with a Normal *p*-Complement 145

8.5	Symmetry of the Group Algebra	147
8.6	Summary of Chapter 8	152
8.7	Exercises for Chapter 8	152
9	**Changing the Ground Ring: Splitting Fields and the Decomposition Map**	158
9.1	Some Definitions	159
9.2	Splitting Fields	159
9.3	The Number of Simple Representations in Positive Characteristic	163
9.4	Reduction Modulo p and the Decomposition Map	167
9.5	The cde Triangle	177
9.6	Blocks of Defect Zero	182
9.7	Summary of Chapter 9	186
9.8	Exercises for Chapter 9	187
10	**Brauer Characters**	192
10.1	The Definition of Brauer Characters	192
10.2	Orthogonality Relations and Grothendieck Groups	198
10.3	The cde Triangle in Terms of Brauer Characters	205
10.4	Summary of Chapter 10	208
10.5	Exercises for Chapter 10	208
11	**Indecomposable Modules**	213
11.1	Indecomposable Modules, Local Rings, and the Krull–Schmidt Theorem	214
11.2	Groups with a Normal Cyclic Sylow p-Subgroup	218
11.3	Relative Projectivity	219
11.4	Finite Representation Type	228
11.5	Infinite Representation Type and the Representations of $C_2 \times C_2$	232
11.6	Vertices, Sources, and Green Correspondence	237
11.7	The Heller Operator	245
11.8	Some Further Techniques with Indecomposable Modules	248
11.9	Summary of Chapter 11	249
11.10	Exercises for Chapter 11	250
12	**Blocks**	257
12.1	Blocks of Rings in General	258
12.2	p-Blocks of Groups	262
12.3	The Defect of a Block: Module Theoretic Methods	266

12.4 The Defect of a Block: Ring Theoretic Methods 270
12.5 The Brauer Morphism 276
12.6 Brauer Correspondence 282
12.7 Further Reading 286
12.8 Summary of Chapter 12 286
12.9 Exercises for Chapter 12 287

Appendix A: Discrete Valuation Rings 291

Appendix B: Character Tables 297

Bibliography 321
Index 323

Preface

The representation theory of finite groups has a long history, going back to the nineteenth century and earlier. A milestone in the subject was the definition of characters of finite groups by Frobenius in 1896. Prior to this there was some use of the ideas that we can now identify as representation theory (characters of cyclic groups as used by number theorists, the work of Schönflies, Fedorov, and others on crystallographic groups, invariant theory, for instance), and during the twentieth century, there was continuously active development of the subject. Nevertheless, the theory of complex characters of finite groups, with its theorem of semisimplicity and the orthogonality relations, is a stunning achievement that remains a cornerstone of the subject. It is probably what many people think of first when they think of finite group representation theory.

This book is about character theory, and it is also about other things: the character theory of Frobenius occupies less than one-third of the text. The rest of the book comes about because we allow representations over rings other than fields of characteristic zero. The theory becomes more complicated, and also extremely interesting, when we consider representations over fields of characteristic dividing the group order. It becomes still more complicated over rings of higher Krull-dimension, such as rings of integers. An important case is the theory over a discrete valuation ring, because this provides the connection between representations in characteristic zero and in positive characteristic. We describe these things in this text.

Why should we want to know about representations over rings that are not fields of characteristic zero? It is because they arise in many parts of mathematics. Group representations appear any time we have a group of symmetries where there is some linear structure present, over some commutative ring. That ring need not be a field of characteristic zero. Here are some examples:

- In number theory, groups arise as Galois groups of field extensions, giving rise not only to representations over the ground field but also to integral representations over rings of integers (in case the fields are number fields). It is natural to reduce these representations modulo a prime ideal, at which point we have modular representations.
- In the theory of error-correcting codes, many important codes have a nontrivial symmetry group and are vector spaces over a finite field, thereby providing a representation of the group over that field.
- In combinatorics, an active topic is to obtain "q-analogs" of enumerative results, exemplified by replacing binomial coefficients (which count subsets of a set) by q-binomial coefficients (which count subspaces of vector spaces over $\mathbb{F}_q$). Structures permuted by a symmetric group are replaced by linear structures acted on by a general linear group, thereby giving representations in positive characteristic.
- In topology, a group may act as a group of self-equivalences of a topological space. Thereby, giving representations of the group on the homology groups of the space. If there is torsion in the homology, these representations require something other than ordinary character theory to be understood.

This book is written for students who are studying finite group representation theory beyond the level of a first course in abstract algebra. It has arisen out of notes for courses given at the second-year graduate level at the University of Minnesota. My aim has been to write the book for the course. It means that the level of exposition is appropriate for such students, with explanations that are intended to be full but not overly lengthy.

Most students who attend an advanced course in group representation theory do not go on to be specialists in the subject, for otherwise the class would be much smaller. Their main interests may be in other areas of mathematics, such as combinatorics, topology, number theory, or commutative algebra. These students need a solid, comprehensive grounding in representation theory that enables them to apply the theory to their own situations as the occasion demands. They need to be able to work with complex characters, and they also need to be able to say something about representations over other fields and rings. While they need the theory to be able to do this, they do not need to be presented with overly deep material whose main function is to serve the internal workings of the subject.

With these goals in mind, I have made a choice of material covered. My main criterion has been to ask whether a topic is useful outside the strict confines of representation theory and, if it is, to include it. At the same time, if there is a theorem that fails the test, I have left it out or put it in the exercises. I have sometimes omitted standard results where they appear not to have sufficiently compelling applications. For example, the theorem of Frobenius on Frobenius groups does not appear, because I do not consider that we need this theorem to understand these groups at the level of this text. I have also omitted Brauer's characterization of characters, leading to the determination of a minimal splitting field for a group and its subgroups. That result is stated without proof, and we do prove what is needed, namely that there exists a finite degree field extension that is a splitting field. For the students who go on to be specialists in representation theory there is no shortage of more advanced monographs. They can find these results there—but they may also find it helpful to start with this book! One of my aims has been to make it possible to read this book from the beginning without having to wade through chapters full of preliminary technicalities, and omitting some results aids in this.

I have included many exercises at the ends of the chapters, and they form an important part of this book. The benefit of learning actively by having to apply the theory to calculate with examples and solve problems cannot be overestimated. Some of these exercises are easy, some more challenging. In a number of instances, I use the exercises as a place to present extensions of results that appear in the text or as an indication of what can be done further.

I have assumed that the reader is familiar with the first properties of groups, rings, and field extensions and with linear algebra. More specifically the reader should know about Sylow subgroups, solvable, and nilpotent groups, as well as the examples that are introduced in a first group theory course, such as the dihedral, symmetric, alternating, and quaternion groups. The reader should also be familiar with tensor products, Noetherian properties of commutative rings, the structure of modules over a principal ideal domain, and the first properties of ideals as well as with Jordan and rational canonical forms for matrices. These topics are covered in a standard graduate-level algebra course. I develop the properties of algebraic integers, valuation theory, and completions within the text since they usually fall outside such a course.

Many people have read sections of this book, worked through the exercises, and been very generous with the comments they have made. I wish to thank them all. They include Cihan Bahran, Dave Benson, Daniel Hess, John Palmieri, Sverre Smalø, and many others.