Probability on Trees and Networks

Starting around the late 1950s, several research communities began relating the geometry of graphs to stochastic processes on these graphs. This book, twenty years in the making, ties together research in the field, encompassing work on percolation, isoperimetric inequalities, eigenvalues, transition probabilities, and random walks. Written by two leading researchers, the text emphasizes intuition, while giving complete proofs and more than 850 exercises. Many recent developments, in which the authors have played a leading role, are discussed, including percolation on trees and Cayley graphs, uniform spanning forests, the mass-transport technique, and connections on random walks on graphs to embedding in Hilbert space.

This state-of-the-art account of probability on networks will be indispensable for graduate students and researchers alike.

RUSSELL LYONS is James H. Rudy Professor of Mathematics at Indiana University, Bloomington. He obtained his PhD at the University of Michigan in 1983. He has written seminal papers concerning probability on trees and random spanning trees in networks. Lyons was a Sloan Foundation Fellow and has been an Invited Speaker at the International Congress of Mathematicians and the Joint Mathematics Meetings. He is a Fellow of the American Mathematical Society.

YUVAL PERES is a Principal Researcher at Microsoft Research in Redmond, Washington. He obtained his PhD at the Hebrew University, Jerusalem in 1990 and later served on their faculty as well as on the faculty at the University of California, Berkeley. He has written more than 250 research papers in probability, ergodic theory, analysis, and theoretical computer science. He has coauthored books on Brownian motion and Markov chain mixing times. Peres was awarded the Rollo Davidson Prize in 1995, the Loève Prize in 2001, and the David P. Robbins Prize in 2011 and was an Invited Speaker at the 2002 ICM.

A love and respect of trees has been characteristic of mankind since the beginning of human evolution. Instinctively, we understood the importance of trees to our lives before we were able to ascribe reasons for our dependence on them.

– James and Louise Bush-Brown, *America's Garden Book*

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Probability on Trees and Networks

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Preface

This book is concerned with certain aspects of discrete probability on infinite graphs that are currently in vigorous development. Of course, finite graphs are analyzed as well, but usually with the aim of understanding infinite graphs and networks. These areas of discrete probability are full of interesting, beautiful, and surprising results, many of which connect to other areas of mathematics and theoretical computer science. Numerous fascinating questions are still open.

Our major topics include random walks and their intimate connection to electrical networks; uniform spanning trees, their limiting forests, and their marvelous relationships with random walks and electrical networks; branching processes; percolation and the powerful, elegant mass-transport technique; isoperimetric inequalities and how they relate to both random walks and percolation; minimal spanning trees and forests and their connections to percolation; Hausdorff dimension, capacity, and how to understand them via trees; and random walks on Galton-Watson trees. Connections among our topics are pervasive and rich, making for surprising and enjoyable proofs.

There are three main classes of graphs on which discrete probability is most interesting, namely, trees, Cayley graphs of groups (or, more generally, transitive, or even quasi-transitive, graphs), and planar graphs. More classical discrete probability has tended to focus on the special and important case of the Euclidean lattices, $\mathbb{Z}^d$, which are prototypical Cayley graphs. This book develops the general theory of various probabilistic processes on graphs and then specializes to the three broad classes listed, always seeing what we can say in the case of $\mathbb{Z}^d$.

Besides their intrinsic interest, there are several reasons for a special study of trees. Since in most cases, analysis is easier on trees, analysis can be carried further. Then one can often either apply the results from trees to other situations or transfer to other situations the techniques developed by working on trees. Trees also occur naturally in many situations, either combinatorially or as descriptions of compact sets in Euclidean space, $\mathbb{R}^d$.

In choosing our topics, we have been swayed by those results we find most striking as well as by those that do not require extensive background. Thus, the only prerequisite is basic knowledge of Markov chains and conditional expectation with respect to a σ-algebra. For Chapter 17, basic knowledge of ergodic theory is also required, though we review it there. Of course, we are highly biased by our own research interests and knowledge. We include the best proofs available of recent as well as classic results.

Most exercises that appear in the text, as opposed to those at the ends of the chapters, are ones that will be particularly helpful to do when they are reached. They either facilitate one's understanding or will be used later in the text. These in-text exercises are also collected at the end of each chapter for easy reference, just before additional exercises are presented. In each chapter, the additional exercises appear in the order that the corresponding material appears in the text.

Some general notation we use is $\langle \cdots \rangle$ for a sequence (or, sometimes, more general function), $\upharpoonright$ for the restriction of a function or measure to a set, $\mathbf{E}[X\,;\ A]$ for the expectation of X on the event A, and $|\bullet|$ for the cardinality of a set. Also, "decreasing" will mean "nonincreasing" unless we say "strictly decreasing," and likewise "increasing" will mean "nondecreasing." Defined terms are in ***bold italics***. Some definitions are repeated in different chapters to enable more selective reading.

A question labeled as **Question** $m.n$ is one to which the answer is unknown, where m and n are numbers. Unattributed results are usually not due to us. Items such as theorems are numbered in this book as $C.n$, where C is the chapter number and n is the item number in that chapter.

Major chapter dependencies are indicated in the first color plate. The plate section, which is between the preface and the first chapter, contains color figures that appear in text as grayscale. Such a figure in text is indicated by the words $^{\text{color}}_{\text{plate}}$ preceding its caption.

It is possible to choose only small parts of various chapters to make a coherent course on specific subjects. For example, a judicious choice of material from the following sections can be used for a one-semester course on relationships of probability to geometric group theory: 3.4, 7.1, 6.1–3, 6.6, 6.7, 13.1–2, 14.1–4, 5.1, 7.2–7, 8.1, 8.3, 8.4, 11.1–4, 11.6, 2.1–5, 6.9, 4.1, 4.2, 9.1, 9.3, 9.4, 10.1, 10.2, 10.9.

In the electronic version of this book, most symbols that are used with a fixed meaning are hyperlinked to their definitions, although the fact that such hyperlinks exist is not made visible.

Many exercises at varying levels of difficulty are included, with many comments, hints, or solutions in the back of the book.

This book began as lecture notes for an advanced graduate course called "Probability on Trees" that Lyons gave in Spring 1993, but its emphasis has been transformed over the intervening years. We are grateful to Rabi Bhattacharya for having suggested that he teach such a course. We have attempted to preserve the informal flavor of lectures.

After Peres joined as a coauthor, writing and research became intertwined, and many delays ensued. Over the course of many months together in Jerusalem, Berkeley, and Redmond, the authors planned the content of most chapters, but the great majority of the actual writing was done by Lyons. Exceptions include especially Chapters 13 and 14 as well as a few sections of other chapters that were mostly written by Peres. Several chapters are based on joint works with Itai Benjamini, Robin Pemantle, and Oded Schramm. A few of the authors' new results appear here for the first time; they are due to both authors in about equal measure. Lyons was responsible for all other aspects of authorship of the book, such as drawing figures, preparing the index, ensuring consistent notation, and typography; most remaining errors can be attributed to him.

Lyons is grateful to the Institute for Advanced Studies and the Institute of Mathematics, both at the Hebrew University of Jerusalem, and to Microsoft Research for support during some of the writing. We are grateful to Brian Barker, Jochen Geiger, Janko Gravner, Yiping (Kenneth) Hu, Svante Janson, Tri Minh Lai, Steve Morrow, Peter Mörters, Minwoo Park, Perla Sousi, Jason Schweinsberg, Jeff Steif, Pengfei Tang, and Ádám Timár for noting several corrections to the manuscript. Section 6.6 and much of Chapter 13 are based on lectures that Peres gave in Berkeley, which were scribed by Asaf Nachmias. In addition, Gábor Pete helped with editing a few sections and provided a careful reading and thoughtful comments throughout. Special thanks are due to Jacob Magnusson for his very thorough and careful reading, which uncovered many small mistakes and possible improvements.

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