Introduction to Modern Magnetohydrodynamics

Ninety-nine percent of ordinary matter in the Universe is in the form of ionized fluids, or plasmas. The study of the magnetic properties of such electrically conducting fluids, magnetohydrodynamics (MHD), has become a central theory in astrophysics, as well as in areas such as engineering and geophysics. This textbook offers a comprehensive introduction to MHD and its recent applications, in nature and in laboratory plasmas; from the machinery of the Sun and galaxies, to the cooling of nuclear reactors and the geodynamo. It exposes advanced undergraduate and graduate students to both classical and modern concepts, making them aware of current research and the ever-widening scope of MHD. Rigorous derivations within the text, supplemented by over 100 illustrations and followed by exercises and worked solutions at the end of each chapter, provide an engaging and practical introduction to the subject and an accessible route into this wide-ranging field.

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To my family

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Preface

Physical laws should have
mathematical beauty
P. A. M. Dirac – Nobel Prize in Physics (1933)

In our familiar environment, matter appears in solid, liquid, or gaseous form. This triptych vision of the world was shaken in the twentieth century when astronomers revealed that most of the extraterrestrial matter – namely more than 99% of the ordinary matter in the Universe – is actually in an ionized state called plasma whose physical properties differ fundamentally from those of a neutral gas. The study of this fourth state of matter was developed mainly in the second half of the twentieth century and is now considered a major branch of modern physics. A decisive step was taken in 1942 when the Swedish astrophysicist Hannes Alfvén (1908–1995) proposed the theory of magnetohydrodynamics (MHD) by connecting the Maxwell electrodynamics with the Navier–Stokes hydrodynamics. In this framework, plasmas are described macroscopically as a fluid and the corpuscular aspect of ions and electrons is ignored. Nowadays, MHD has emerged as the central theory to understand the machinery of the Sun, stars, stellar winds, accretion disks around super-massive objects such as black holes with the formation of extragalactic jets, interstellar clouds, and planetary magnetospheres. Also, when H. Alfvén was awarded the Nobel Prize in Physics in 1970, the Committee congratulated him "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics."

The MHD description is not limited to astrophysical plasmas, but is also widely used in the framework of laboratory experiments or industrial developments for which plasmas and conducting liquid metals are used. In the first case, the emblematic example is certainly controlled nuclear fusion with the International Thermonuclear Experimental Reactor (ITER) in Cadarache. Indeed, the control of a magnetically confined plasma requires an understanding of the large-scale equilibrium and the solution of stability problems whose theoretical framework is basically MHD. Liquid metals are also used, for example, in experiments to investigate the mechanism of magnetic field generation – the dynamo effect – that occurs naturally in the liquid outer core of our planet via turbulent motions of a mixture of liquid metals. Most of the natural MHD flows cited above are far from thermodynamic equilibrium, with highly turbulent dynamics. Furthermore, a finer description including the most important effect, i.e. the decoupling effect between the ions and the electrons – the Hall effect, is nowadays often used to understand observations and experiments. Thus, an introduction to modern MHD must include both turbulence and the Hall effect, which is the case of this book where a systematic comparison with recent research is made with a large number of citations.

This textbook is an introduction to modern MHD. It provides a clear connection between the theory and recent experimental results. It aims at presenting the main physical properties and applications of plasmas or liquid MHD flows starting from the knowledge of an undergraduate student. It is therefore addressed primarily to advanced undergraduate students, postgraduate (Masters) students – regardless of their area of specialization (astrophysics, plasma, fusion, or fluid mechanics), and engineering students wishing to complete their training. Mathematical derivations are rigorous and the results are illustrated with more than 100 figures, some of which originate from the most recent experimental measurements. Exercises with their solutions complete the presentation. Approximately 80% of the content of this textbook corresponds to a one-semester postgraduate MHD course that I give regularly at the Université Paris–Saclay and which was published in French in 2013. The present version is its English translation with some new material.

I am grateful to all my Masters students, PhD students, and colleagues with whom I have discussed MHD, and in particular to Supratik Banerjee, Romain Meyrand, and Caroline Nore.

Paris, 29 August 2015 Sébastien Galtier

Table of Physical Quantities

Numerical values of some (plasma) parameters appearing in the main text. The international system (IS) is used (densities n_e and n_i are in m^{-3}; magnetic field B in tesla (T); temperature T in kelvins; magnitude of the electron charge e in coulombs; electron and ions masses m_e and m_i in kg) and we assume that ions are only protons. In the evaluations of ν and η, we consider a completely ionized plasma (Spitzer 1962).

Electron plasma frequency	$\frac{\omega_{pe}}{2\pi} = \frac{(n_\mathrm{e} e^2/m_\mathrm{e}\varepsilon_0)^{1/2}}{2\pi} \simeq 8.98\, n_\mathrm{e}^{1/2}$ Hz
Ion plasma frequency	$\frac{\omega_{pi}}{2\pi} = \frac{(n_\mathrm{i} e^2/m_\mathrm{i}\varepsilon_0)^{1/2}}{2\pi} \simeq 0.21\, n_\mathrm{i}^{1/2}$ Hz
Electron inertial length	$d_\mathrm{e} = c/\omega_\mathrm{pe} \simeq 5.3 \times 10^6\, n_\mathrm{e}^{-1/2}$ m
Ion inertial length	$d_\mathrm{i} = c/\omega_\mathrm{pi} \simeq 2.3 \times 10^8\, n_\mathrm{i}^{-1/2}$ m
Electron gyrofrequency	$\frac{\omega_\mathrm{ce}}{2\pi} = \frac{eB/m_\mathrm{e}}{2\pi} \simeq 2.8 \times 10^{10} B$ Hz
Ion gyrofrequency	$\frac{\omega_\mathrm{ci}}{2\pi} = \frac{eB/m_\mathrm{i}}{2\pi} \simeq 1.5 \times 10^{7} B$ Hz
Kinematic viscosity	$\nu \simeq 10^{10}\, T^{5/2}\, n_\mathrm{i}^{-1}$ m^2/s
Magnetic diffusivity	$\eta \simeq 10^{9}\, T^{-3/2}$ m^2/s
Reynolds number	$R_\mathrm{e} = Lu/\nu$
Magnetic Reynolds number	$R_\mathrm{m} = Lu/\eta$
Lundquist number	$\mathcal{S} = LB/(\eta\sqrt{\mu_0 m_\mathrm{i} n_\mathrm{i}})$
Magnetic field strength	1 Tesla = 10^4 Gauss
Magnetic pressure	$P_\mathrm{m} = B^2/(2\mu_0) \simeq 4 \times 10^5 B^2$ Pa
Length scales	1 pc $\simeq$ 3.2 light-years $\simeq 3 \times 10^{16}$ m