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Alec J. Schramm
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Mathematical Methods and Physical Insights

An Integrated Approach

Mathematics instruction is often more effective when presented in a physical context. Schramm uses this insight to help develop students' physical intuition as he guides them through the mathematical methods required to study upper-level physics. Based on the undergraduate Math Methods course he has taught for many years at Occidental College, the text encourages a symbiosis through which the physics illuminates the math, which in turn informs the physics. Appropriate for both classroom and self-study use, the text begins with a review of useful techniques to ensure students are comfortable with prerequisite material. It then moves on to cover vector fields, analytic functions, linear algebra, function spaces, and differential equations. Written in an informal and engaging style, it also includes short supplementary digressions ("By the Ways") as optional boxes showcasing directions in which the math or physics may be explored further. Extensive problems are included throughout, many taking advantage of Mathematica, to test and deepen comprehension.

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To Eitan, whose wit and humor eases my load
To Tirtza, who leads by example to show me the way
To Aviya, who teaches that obstacles need not block my road
To Keshet, whose refracted sunshine colors each day

And to Laurel, whose smile lights my life

Contents

List of BTWs

Preface

You're holding in your hands a book on mathematical methods in the physical sciences. Wait — don't put it down just yet! I know, I know: there are already many such books on the shelves of scientists and engineers. So why another one?

Motivations

As with many textbooks, this one has its origins in the classroom — specifically, a math methods course I teach annually for third-year physics and engineering students. The project began, simply enough, with the realization that many students have trouble integrating functions beyond polynomials and elementary trig. At first I thought this merely reflected gaps in their background. After all, it's not unusual for standard calculus courses to largely skip once-invaluable topics such as trig substitution. To be sure, there's a cogent argument to be made that many of the integration techniques taught back in my college days are passé in the age of such resources as Wolfram Alpha. One unfortunate consequence, however, is that students too often see an integral as a black-box calculator, rather than as a mathematical statement in its own right. And indeed, I soon realized that the difficulties I was encountering had less to do with students' integration abilities and more with their basic comprehension. So one year I wrote a handout for the first week of the semester, in the hope that learning a few standard techniques would help demystify integration. I was astounded by the overwhelmingly positive response. Moreover, despite the relatively quick pace, students became notably less intimidated by integrals common in upper-level physics and engineering. Though they did not always master the techniques, most students began to develop a solid intuition for both the physics and mathematics involved — which was the goal, after all. This early handout evolved into Chapter 8, "Ten Integration Techniques and Tricks."

Encouraged by this modest success — and having identified other areas of similar weakness — I began to develop a set of notes on an array of topics. Although upper-level students have already seen most of this material in other courses, I've found that even if they can push the symbols around, their understanding is often wanting. And once again, the results were gratifying. These notes form the backbone of Part I of this book, which is intended as a review with an eye toward strengthening intuition.

As I continued writing handouts for various topics covered in the course, a book began to emerge. Along the way I found myself influenced by and reacting to the many texts which often stress formality over pragmatism, or are too recipe-driven to instill insight. Neither approach, in my view, gets it quite right. All too often, students lost in the abstract find refuge in rote symbol manipulation; emphasizing manipulation alone, however, is woefully inadequate. The ability to play the notes doesn't make you a musician.[1] So it's a delicate balance. An effective program must be more than a collection of theorems, proofs, or How-To's, and should be illustrated with physical examples whose context is familiar to the intended audience.

The approach taken here is not meant to be rigorous; indeed, many of us were attracted to the physical sciences more by the playfulness and fun of mathematics than its formality. For the most part, derivations and proofs are reserved for situations in which they aid comprehension. My goal has been to leverage that delight we've all experienced when the depth and inherent beauty of a mathematical concept or technique reveal themselves — not just the endorphin rush of understanding, but the empowering command that comes with intuition.

Content and Organization

Of course, one has to start somewhere. It's impractical to go all the way back to an introduction of slopes and derivatives, or the development of the integral from a Riemann sum. For such topics, many excellent texts are readily available. As this book is aimed at upper-level physics and engineering students, the presumed background includes standard introductory physics through the sophomore level. A background in multivariable calculus and linear algebra is also assumed — though some of this material is reviewed in Parts II and IV.

Much of the coverage is similar to that of other math methods texts (as indeed it must be), so there's more material than is generally covered in two semesters.[2] But the design is intended to give instructors flexibility without sacrificing coherence. The book is separated into Parts. Each begins with a prelude introduction, has an interlude to try to flesh out intuition, and ends with a coda to tie together many of the ideas in a physical context. Some of the interludes and codas can be assigned as reading rather than presented during class time without loss of continuity. Aside from Part I (see below), the Parts were constructed to build upon one another pedagogically, with properties and representations of vectors as a common theme. Even so, the Parts are relatively self-contained.

- Part I: With rare exception, instructors of math methods courses cite the same central challenge: the uneven preparation of students. This creates an enormous pedagogical hurdle on both sides of the white board. Addressing this obstacle is the motivation underlying Part I, which, as its title suggests, students Just Gotta' Know.[3] I've found that starting the semester with a potpourri of topics helps smooth out students' variation in background. In fact, despite the unit's staccato character, students across the preparation spectrum have clearly derived benefit — as well as, I daresay, a sense of empowerment. I usually cover this material over the first 2–3 weeks of the semester, though one could envision covering some sections only as needed as the term progresses.
- Part II: This unit is largely an introduction to grad, div, and curl. For completeness, a review of general line and surface integrals is given — but even here, the goal is to prepare for the theorems of Gauss and Stokes. The primary paradigm is the behavior of electric and magnetic fields. Though the assumption is that students will have had a typical first-year E&M course, I've tried to make these examples largely self-contained. The Coda discusses simply connected regions, and includes an example of a circuit with non-trivial topology.

[1] As I often ask my students, would you want to fly on a plane designed by someone who knows how to push around symbols, but with a weak grasp of their meaning? I'm reminded of my 7th grade math teacher, Mr. Ford, who used to berate students with the line "It's not for you to understand — it's for you to do!" I cannot in good conscience recommend his airplane designs.

[2] Good general references include Arfken and Weber (1995); Boas (2005); Kusse and Westwig (1998); Mathews and Walker (1970), and Wong (1991).

[3] Because of its potpourri nature, I wanted to call it "Unit 0," but my editors wisely advised that this would be seen as too quirky.

- Part III: As a natural extension of the discussion in the previous unit, this one begins by reformulating $\mathbb{R}^2$ vector fields as functions in the complex plane. Note that Part I only discusses the rudiments of complex numbers — more algebra than calculus. The topics addressed in this unit include Laurent series, Cauchy's integral formula, and the calculus of residues. The Interlude considers conformal mapping, and the Coda the connection between analyticity and causality. Instructors who prefer to cover these subjects at another time can choose to skip this unit on a first pass, since little of the material is needed later.
- Part IV: This unit turns to linear algebra, and as such the focus shifts from vectors as functions in $\mathbb{R}^2$ and $\mathbb{R}^3$, to basic mathematical entities which do little more than add and subtract. Over the course of the unit more structure is introduced — most notably, the inner product. Complete orthonormal bases are emphasized, and Dirac's bra-ket notation is introduced. Orthogonal, unitary, and Hermitian matrices are discussed. An Interlude on rotations is complemented by an appendix presenting different approaches to rotations in $\mathbb{R}^3$. We then turn to the eigenvalue problem and diagonalization. The Coda focuses on normal modes, serving as a natural bridge to Part V.
- Entr'acte: Following the developments in the previous unit, the Entr'acte classifies vectors in terms of their behavior under rotation. This naturally leads to discussions of tensors, metrics, and general covariance.
- Part V: Here the results of Part IV are extended from finite- to infinite-dimensional vector space. Orthogonal polynomials and Fourier series are derived and developed as examples of complete orthogonal bases. Taking the continuum limit leads to Fourier transforms and convolution, and then to Laplace transforms. The Interlude introduces spherical harmonics and Bessel functions as examples of orthogonal functions beyond sine and cosine — the goal being to develop familiarity and intuition for these bases in advance of tackling their defining differential equations in Part VI. The Coda is an introduction to signal processing, with an emphasis on Fourier analysis and the convolution theorem.
- Part VI: After a relatively brief discussion of first-order differential equations, this unit turns to second-order ODEs and PDEs. The orthogonal polynomials of Part V are rederived by series solutions. Initial and boundary value problems, as well as Green's function solutions are introduced. The unitarity and hermiticity of differential operators are examined in an Interlude on the Sturm–Liouville problem, and the Coda provides an introduction to quantum mechanical scattering as an application of Green's functions.

It's a challenge to bring students up to speed while simultaneously lifting and enticing them to the next level. Towards this end, sprinkled throughout the text are asides called "By The Ways" (BTWs). Averaging about a page in length, these digressions are what I hope are delightful ("geeky-cool") forays into the tangential — sometimes math, sometimes physics. Though the array of BTW topics is necessarily rather idiosyncratic, they were chosen for beauty, charm, or insight — and in the best instances, all three. Since the BTWs are not required for understanding of the mainstream material, students can choose to skip over them (though they'll miss much of the fun).

One of my professors liked to say "The best professor is the worst professor." By this he meant that ultimately one must be self-taught — and that an ineffective lecturer is often required to compel students to learn on their own. Though personally I think he was just making excuses for poor lecture prep, it is true that students cannot master the material without working problems themselves. So each chapter includes extensive homework exercises, many written to take advantage of the capabilities of Mathematica.[4] (Online solutions are available for instructors.) The objective, of course, is to recruit and retain every student's best teacher.

[4] Trott (2004) is a good reference source.

Inspiration and Aspiration

Ultimately, my own experiences as a student of physics and math motivate the guiding principle: the physics illuminates and encourages the math, which in turn further elucidates the physics. But it doesn't come easy; effort is required. As I often tell students, walking through an art museum or listening to music will not elicit the same depth of wonder and delight without having first studied art or music. So too, appreciation of the inherent harmonies between math and physics favors the prepared mind. Moreover, such leveraging of the physics/math symbiosis is truly a life-long exercise. But it is my hope that this approach will enhance students' appreciation for the inherent beauty of this harmony, inspiring them to continue their pursuit into the wonders and mysteries of physical science.

Acknowledgements

I want to begin by thanking my parents, who sadly are no longer here to show this book to their friends. To my brothers, from whom I've learned so much on a remarkably wide array of subjects. And most especially to Laurel and our children, whose love and support I could not do without.

To the incredible professionals Vince Higgs, Heather Brolly, and especially Melissa Shivers at Cambridge University Press, my deep appreciation for their guidance, advice, and seemingly inexhaustible patience. And to David Hemsley, for his sharp eye and good humor. It truly has been a pleasure working and bantering with all of them.

Mr. Ford and the above-referenced professor notwithstanding, I've had the good fortune of learning from some incredible teachers and colleagues, in both formal and informal settings. But I've also learned from students. In particular, I want to thank the many who have taken my courses over the years using early versions of the manuscript. Special shout-outs of gratitude to Olivia Addington, Claire Bernert, Ian Convy, Emily Duong, Alejandro Fernandez, Leah Feuerman, Michael Kwan, Sebastian Salazar, Hunter Weinreb, and Hedda Zhao. Their (mostly constructive) criticisms, suggestions, and perspective have greatly improved the text. Without Jason Detweiler this book may never have seen the light of day. And deepest gratitude and appreciation to my friend and colleague Jochen Rau, who read much of the text, prevented many embarrassing errors — and taught me the word "betriebsblind."

Any remaining errors, of course, are mine alone; they tried their best.