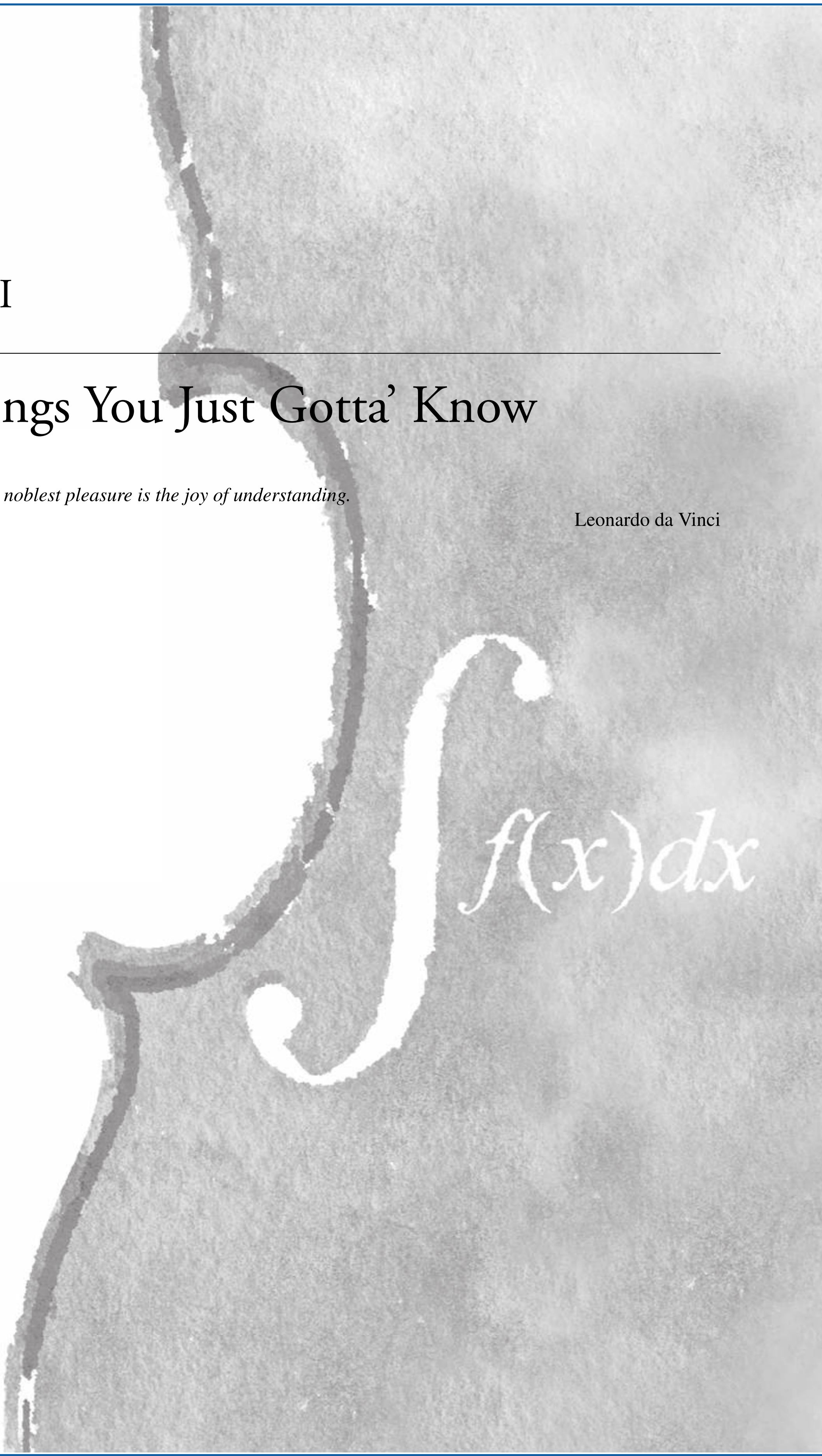
Part I

Things You Just Gotta' Know

The noblest pleasure is the joy of understanding.

Leonardo da Vinci



CAMBRIDGE
Cambridge University Press & Assessment
978-1-107-15641-8 — Mathematical Methods and Physical Insights
Alec J. Schramm
Excerpt
More Information

1 Prelude: Symbiosis

We begin with some basic tools. Many of these will undoubtedly be review; perhaps a few are new. Still other techniques may be familiar but never mastered. Indeed, it is not uncommon to possess an ability to push symbols around without a firm appreciation of their meaning. And sometimes symbol-pushing camouflages that lack of understanding — even to oneself. This seldom ends well.

Moreover, it's important to realize that mathematical concepts and techniques are more than mere tools: they can be a source of physical insight and intuition. Conversely, physical systems often inspire an appreciation for the power and depth of the math — which in turn can further elucidate the physics. This sort of leveraging infuses much of the book. In fact, let's start with three brief illustrations of the symbiotic relationship between math and physics — examples we'll return to often.

Example 1.1 Oscillations

One of the most important equations in physics is

$$\frac{d^2x}{dt^2} = -\omega^2 x, \tag{1.1}$$

which describes simple harmonic motion with frequency $\omega = 2\pi f$. The solution can be written

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{1.2}$$

where the amplitudes of oscillation A and B can be determined from initial values of position and speed. The significance of this equation lies in the ubiquity of oscillatory motion; moreover, many of the most common differential equations in physics and engineering are in some sense generalizations of this simple expression. These "variations on a theme" consider the effects of additional terms in (1.1), resulting in systems rich both mathematically and physically. But even the simple variation

$$\frac{d^2x}{dt^2} = +\omega^2 x \tag{1.3}$$

brings new possibilities and insight. Although differing from (1.1) by a mere sign, this seemingly trivial mathematical distinction yields a solution which is completely different physically:

$$x(t) = Ae^{-\omega t} + Be^{+\omega t}. \tag{1.4}$$

The mathematics is clearly hinting at an underlying relationship between oscillation and exponential growth and decay. Appreciating this deeper connection requires we take a step back to consider *complex numbers*. Given that position $x(t)$ is real, this may seem counterintuitive. But as we'll see in Part I and beyond, the broader perspective provided by allowing for complex numbers generates physical insight.

Example 1.2 Angular Momentum and Moment of Inertia

Recall that angular momentum is given by

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}, \tag{1.5}$$

where the angular velocity $|\vec{\omega}| = v/r$, and I is the moment of inertia. In introductory mechanics, one is told that moment of inertia is a scalar quantity which measures the distribution of mass around the axis of rotation. Thus (1.5) clearly shows that angular momentum $\vec{L}$ is always parallel to the angular velocity $\vec{\omega}$. Right?

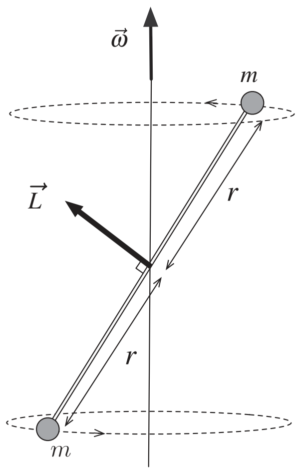


Figure 1.1 Angular momentum of a dumbbell.

Wrong. Perhaps the simplest counter-example is a massless rod with two identical weights at either end, spinning as shown in Figure 1.1 with angular velocity $\vec{\omega}$ pointing vertically (Figure 1.1). With the origin at the center of the rod, each mass contributes equally to the total angular momentum. Since the momentum of each is perpendicular to its position vector $\vec{r}$, the angular momentum is easily found to have magnitude

$$L = |\vec{r} \times \vec{p}| = 2mvr = 2mr^2\omega. \tag{1.6}$$

This much is certainly correct. The problem, however, is that $\vec{L}$ is not parallel to $\vec{\omega}$! Thus either $\vec{L} = I\vec{\omega}$ holds only in special circumstances, or I cannot be a simple scalar. The resolution requires expanding our mathematical horizons into the realm of *tensors* — in which scalars and vectors are special cases. In this unit we'll introduce notation to help manage this additional level of sophistication; a detailed discussion of tensors is presented in the Entr'acte.

Example 1.3 Space, Time, and Spacetime

The Special Theory of Relativity is based upon two postulates put forward by Einstein in 1905:

I. The laws of physics have the same form in all inertial reference frames.
II. The speed of light in vacuum is the same in all inertial reference frames.

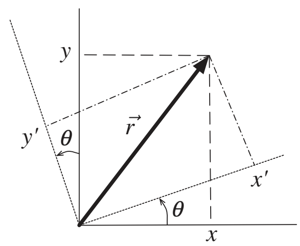
The first postulate is merely a statement of Galileo's principle of relativity; the second is empirical fact. From these postulates, it can be shown that time and coordinate measurements in two frames with relative speed V along their common x-axis are related by the famous *Lorentz transformations*,

$$t' = \gamma(t - Vx/c^2) \qquad y' = y$$
$$x' = \gamma(x - Vt) \qquad z' = z, \tag{1.7}$$

where $\gamma \equiv (1 - V^2/c^2)^{-1/2}$. At its simplest, special relativity concerns the relationship between measurements in different inertial frames.

Intriguingly, we can render the Lorentz transformations more transparent by rescaling the time as $x_0 \equiv ct$, so that it has the same dimensions as x; the second postulate makes this possible, since it promotes the speed of light to a universal constant of nature. To make the notation consistent, we'll also define $(x_1, x_2, x_3) \equiv (x, y, z)$. Then introducing the dimensionless parameter $\beta \equiv V/c$, (1.7) become

$$x'_0 = \gamma(x_0 - \beta x_1) \qquad x'_2 = x_2$$
$$x'_1 = \gamma(x_1 - \beta x_0) \qquad x'_3 = x_3. \tag{1.8}$$



In this form, the equal footing of space and time, a hallmark of relativity, is manifest. It also clearly displays that transforming from one frame to another "mixes" space and time. Curiously, this is very similar to the way a rotation mixes up position coordinates. Upon rotation of coordinate axes, a vector's new components (x', y') are a simple linear combination of the old components (x, y). Could it be that relativity similarly "rotates" space and time?

In fact, Lorentz transformations can be understood as rotations in a single, unified mathematical structure called *spacetime* — also known as *Minkowski space* after the mathematician who first worked out this geometric interpretation. Recognition of the fundamental nature of spacetime represents one of the greatest paradigm shifts in the history of science: the nearly 300 year-old notion of a three-dimensional universe plus an external time parameter is replaced by a single *four*-dimensional structure. Deeper insight into the mathematics of rotations will heighten the appreciation for this shift.

Cambridge University Press & Assessment
978-1-107-15641-8 — Mathematical Methods and Physical Insights
Alec J. Schramm
Excerpt
More Information

2 Coordinating Coordinates

2.1 Position-Dependent Basis Vectors

Working with a vector-valued function $\vec{V}(\vec{r})$ often involves choosing two coordinate systems: one for the functional dependence, and one for the basis used to decompose the vector. So a vector in the plane, say, can be rendered on the cartesian basis $\{\hat{\imath}, \hat{\jmath}\}$ using either cartesian or polar coordinates,[1]

$$\vec{V}(\vec{r}) = \hat{\imath}\, V_x(x,y) + \hat{\jmath}\, V_y(x,y) \tag{2.1a}$$

$$= \hat{\imath}\, V_x(\rho,\phi) + \hat{\jmath}\, V_y(\rho,\phi), \tag{2.1b}$$

where $V_x \equiv \hat{\imath} \cdot \vec{V}$ and $V_y \equiv \hat{\jmath} \cdot \vec{V}$. But we could just as easily choose the polar basis $\{\hat{\rho}, \hat{\phi}\}$,

$$\vec{V}(\vec{r}) = \hat{\rho}\, V_\rho(\rho,\phi) + \hat{\phi}\, V_\phi(\rho,\phi) \tag{2.1c}$$

$$= \hat{\rho}\, V_\rho(x,y) + \hat{\phi}\, V_\phi(x,y)\,, \tag{2.1d}$$

with $V_\rho \equiv \hat{\rho} \cdot \vec{V}$ and $V_\phi \equiv \hat{\phi} \cdot \vec{V}$. For instance, the position vector $\vec{r}$ can be written

$$\vec{r} = \hat{\imath}\, x + \hat{\jmath}\, y \tag{2.2a}$$

$$= \hat{\imath}\, \rho \cos\phi + \hat{\jmath}\, \rho \sin\phi \tag{2.2b}$$

$$= \hat{\rho}\, \rho \tag{2.2c}$$

$$= \hat{\rho}\, \sqrt{x^2 + y^2}. \tag{2.2d}$$

To date, you have probably had far more experience using cartesian components. At the risk of stating the obvious, this is because the cartesian unit vectors $\hat{\imath}, \hat{\jmath}$ never alter direction — each is parallel to itself everywhere in space. This is not true in general; in other systems *the directions of the basis vectors depend on their spatial location.* For example, at the point $(x,y) = (1,0)$, the unit vectors in polar coordinates are $\hat{\rho} \equiv \hat{\imath}$, $\hat{\phi} \equiv \hat{\jmath}$, whereas at $(0,1)$ they are $\hat{\rho} \equiv \hat{\jmath}$, $\hat{\phi} \equiv -\hat{\imath}$ (Figure 2.1). A position-dependent basis vector is not a trivial thing.

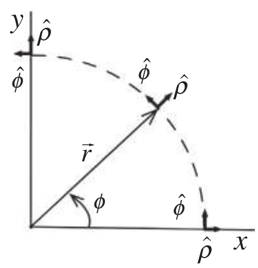


Figure 2.1 The polar basis.

Because the cartesian unit vectors are constant, they can be found by taking derivatives of the position $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ — for instance, $\hat{\jmath} = \partial\vec{r}/\partial y$. We can leverage this to find $\hat{\rho}$ and $\hat{\phi}$ as functions of position, beginning with the chain rule in polar coordinates,

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho}\, d\rho + \frac{\partial \vec{r}}{\partial \phi}\, d\phi. \tag{2.3}$$

[1] $V(x,y)$ is generally a different function than $V(\rho,\phi)$; we use the same symbol for simplicity.

Now if we hold ϕ fixed but vary ρ by $d\rho$, the resulting displacement $d\vec{r}$ must be in the $\hat{\rho}$-direction. In other words, $\partial\vec{r}/\partial\rho$ is a vector tangent to curves of constant ϕ. Similarly, $\partial\vec{r}/\partial\phi$ is a vector tangent to curves of constant ρ, and points in the $\hat{\phi}$-direction. (See Problem 6.36.) To calculate unit tangent vectors, we decompose the position vector $\vec{r}$ along cartesian axes but with components expressed in polar coordinates,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = \rho\cos\phi\,\hat{\imath} + \rho\sin\phi\,\hat{\jmath}. \tag{2.4}$$

Then we find

$$\hat{\rho} \equiv \frac{\partial\vec{r}/\partial\rho}{|\partial\vec{r}/\partial\rho|} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath} \tag{2.5a}$$

and

$$\hat{\phi} \equiv \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}\,. \tag{2.5b}$$

Note that the fixed nature of the cartesian basis is crucial in this derivation. As a quick consistency check, not only are these vectors normalized, $\hat{\rho}\cdot\hat{\rho} = \hat{\phi}\cdot\hat{\phi} = 1$, they are also orthogonal, $\hat{\rho}\cdot\hat{\phi} = 0$. And as expected, they are position-dependent; direct differentiation of (2.5) gives

$$\frac{\partial\hat{\rho}}{\partial\phi} = \hat{\phi} \qquad \frac{\partial\hat{\phi}}{\partial\phi} = -\hat{\rho}, \tag{2.6a}$$

and also, as expected,

$$\frac{\partial\hat{\rho}}{\partial\rho} = 0 \qquad \frac{\partial\hat{\phi}}{\partial\rho} = 0. \tag{2.6b}$$

We can trivially extend these results to cylindrical coordinates (ρ,ϕ,z), since $\hat{k} \equiv \hat{z}$ is a constant basis vector.

Example 2.1 Velocity in the Plane

Consider the velocity $\vec{v} = d\vec{r}/dt$. Along cartesian axes, this is just

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{\imath}) + \frac{d}{dt}(y\hat{\jmath}) = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath}\,, \tag{2.7}$$

while on the polar basis we have instead

$$\begin{aligned} \frac{d\vec{r}}{dt} = \frac{d}{dt}(\rho\hat{\rho}) &= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} \\ &= \frac{d\rho}{dt}\hat{\rho} + \rho\left(\frac{\partial\hat{\rho}}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\hat{\rho}}{\partial\phi}\frac{d\phi}{dt}\right), \end{aligned} \tag{2.8}$$

where the chain rule was used in the last step. Of course, (2.7) and (2.8) are equivalent expressions for the same velocity. As it stands, however, (2.8) is not in a particularly useful form: if we're going to use the polar basis, $\vec{v}$ should be clearly expressed in terms of $\hat{\rho}$ and $\hat{\phi}$, not their derivatives.

This is where (2.6) come in, allowing us to completely decompose the velocity on the polar basis:

$$\begin{aligned} \vec{v} &= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\phi}{dt}\frac{\partial\hat{\rho}}{\partial\phi} \\ &\equiv \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi}, \end{aligned} \tag{2.9}$$

where we have used the common "dot" notation to denote time derivative, $\dot{u} \equiv du/dt$. The forms of the radial and tangential components,

$$v_\rho = \dot{\rho}, \qquad v_\phi = \rho\dot{\phi} \equiv \rho\,\omega, \tag{2.10}$$

should be familiar; $\omega \equiv \dot{\phi}$, of course, is the angular velocity.

Example 2.2 Acceleration in the Plane

In cartesian coordinates, a particle in the plane has acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}, \tag{2.11}$$

where $\ddot{u} \equiv d^2u/dt^2$. The same calculation in polar coordinates is more complicated, but can be accomplished without difficulty: differentiating (2.9) and using (2.6a) and (2.6b) gives

$$\begin{aligned}\vec{a} &= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi} \\ &\equiv \left(\ddot{\rho} - \rho\omega^2\right)\hat{\rho} + \left(\rho\alpha + 2\dot{\rho}\omega\right)\hat{\phi},\end{aligned} \tag{2.12}$$

where $\alpha \equiv \ddot{\phi}$ is the angular acceleration. The $\ddot{\rho}$ and $\rho\alpha$ terms have straightforward interpretations: changing speed in either the radial or tangential directions results in acceleration. The $-\rho\omega^2\hat{\rho}$ term is none other than centripetal acceleration (recall that $\rho\omega^2 = v_\phi^2/\rho$); herein is the elementary result that acceleration can occur even at constant speed. The fourth term, $2\dot{\rho}\omega\hat{\phi}$, is the Coriolis acceleration observed within a non-inertial frame rotating with angular velocity $\omega\,\hat{\phi}$.

Though a little more effort, the procedure that led to (2.6a) and (2.6b) also works for spherical coordinates r, θ, and ϕ. Starting with

$$d\vec{r} = \frac{\partial\vec{r}}{\partial r}\,dr + \frac{\partial\vec{r}}{\partial\theta}\,d\theta + \frac{\partial\vec{r}}{\partial\phi}\,d\phi \tag{2.13}$$

and

$$\vec{r} = r\,\sin\theta\,\cos\phi\,\hat{\imath} + r\,\sin\theta\,\sin\phi\,\hat{\jmath} + r\,\cos\theta\,\hat{k}, \tag{2.14}$$

we use the tangent vectors to define

$$\hat{r} \equiv \frac{\partial\vec{r}/\partial r}{|\partial\vec{r}/\partial r|} \qquad \hat{\theta} \equiv \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} \qquad \hat{\phi} \equiv \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|}. \tag{2.15}$$

As you'll verify in Problem 2.1, this leads to

$$\hat{r} = \hat{\imath}\sin\theta\cos\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{2.16a}$$

$$\hat{\theta} = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta \tag{2.16b}$$

$$\hat{\phi} = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi, \tag{2.16c}$$

so that

$$\partial\hat{r}/\partial\theta = \hat{\theta} \qquad \partial\hat{r}/\partial\phi = \sin\theta\,\hat{\phi} \tag{2.17a}$$

$$\partial\hat{\theta}/\partial\theta = -\hat{r} \qquad \partial\hat{\theta}/\partial\phi = \cos\theta\,\hat{\phi} \tag{2.17b}$$

$$\partial\hat{\phi}/\partial\theta = 0 \qquad \partial\hat{\phi}/\partial\phi = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}. \tag{2.17c}$$

And of course, all the unit vectors have vanishing radial derivatives.

Once again, a quick consistency check verifies that $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ form an *orthonormal* system — that is, that they are unit vectors ("normalized"),

$$\hat{r} \cdot \hat{r} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1,$$

and they're mutually orthogonal,

$$\hat{r} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{r} = 0.$$

Moreover, the so-called "right-handedness" of cartesian coordinates, $\hat{\imath} \times \hat{\jmath} = \hat{k}$, bequeathes an orientation to spherical coordinates,[2]

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \Longleftrightarrow \hat{r} \times \hat{\theta} = \hat{\phi}.$$

Similarly, in cylindrical coordinates $\hat{\rho} \times \hat{\phi} = \hat{k}$. (Caution: ρ denotes the cylindrical radial coordinate, r the spherical. But it's very common to use r for both — as we will often do. So pay attention to context!)

Example 2.3 A First Look at Rotations

There's a simple and handy way to convert back and forth between these coordinate bases. A mere glance at (2.5) and (2.16) reveals that they can be expressed in matrix form,

$$\begin{pmatrix} \hat{\rho} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \end{pmatrix}, \tag{2.18}$$

and

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix}. \tag{2.19}$$

The inverse transformations $\{\hat{\rho}, \hat{\phi}\} \to \{\hat{\imath}, \hat{\jmath}\}$ and $\{\hat{r}, \hat{\theta}, \hat{\phi}\} \to \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ can be found either by inverting (2.5) and (2.16) algebraically, or equivalently by finding the inverse of the matrices in (2.18) and (2.19). But this turns out to be very easy, since (as you can quickly verify) the inverses are just their transposes. Thus

$$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\phi} \end{pmatrix}, \tag{2.20}$$

and

$$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}. \tag{2.21}$$

In Part III, we'll see that $R^{-1} = R^T$ is a defining feature of rotations in $\mathbb{R}^n$.

2.2 Scale Factors and Jacobians

As a vector, $d\vec{r}$ can be decomposed on any basis. As you'll show in Problem 2.3, the chain rule — together with (2.4) and (2.5) — leads to the familiar line element in cylindrical coordinates,

$$d\vec{r} = \hat{\rho}\, d\rho + \hat{\phi}\, \rho\, d\phi + \hat{k}\, dz. \tag{2.22}$$

[2] A left-handed system has $\hat{\jmath} \times \hat{\imath} = \hat{k}$ and $\hat{\theta} \times \hat{r} = \hat{\phi}$.

The extra factor of ρ in the second term is an example of a *scale factor*, and has a simple geometric meaning: at a distance ρ from the origin, a change of angle $d\phi$ leads to a change of distance $\rho\, d\phi$. Such scale factors, usually denoted by h, neatly summarize the geometry of the coordinate system. In cylindrical coordinates,

$$h_\rho \equiv 1, \qquad h_\phi \equiv \rho, \qquad h_z \equiv 1. \tag{2.23}$$

In spherical, the scale factors are

$$h_r \equiv 1, \qquad h_\theta \equiv r, \qquad h_\phi \equiv r\sin\theta, \tag{2.24}$$

corresponding to the line element

$$d\vec{r} = \hat{r}\, dr + \hat{\theta}\, r\, d\theta + \hat{\phi}\, r\sin\theta\, d\phi. \tag{2.25}$$

So in spherical coordinates, a change of $d\theta$ does not by itself give the displacement; for that we need an additional factor of r. Indeed, neither $d\theta$ nor $d\phi$ has the same units as $d\vec{r}$; scale factors, which are generally functions of position, remedy this.

BTW 2.1 Curvilinear Coordinates I

Cylindrical and spherical coordinates are the two most common $\mathbb{R}^3$ examples of *curvilinear coordinates*. Cartesian coordinates, of course, are notable by their straight axes and constant unit vectors; by contrast, curvilinear systems have curved "axes" and position-dependent unit vectors. The procedure we used to derive the cylindrical and spherical unit vectors and scale factors can be generalized to any set of curvilinear coordinates.

Let's simplify the notation by denoting cartesian coordinates as x_i, where $i = 1, 2$, or 3; for curvilinear coordinates, we'll use u_i with unit vectors $\hat{e}_i$. (So in cylindrical coordinates, we have the assignments $\hat{e}_1 \to \hat{\rho}$, $\hat{e}_2 \to \hat{\phi}$, $\hat{e}_3 \to \hat{k}$, and in spherical $\hat{e}_1 \to \hat{r}$, $\hat{e}_2 \to \hat{\theta}$, $\hat{e}_3 \to \hat{\phi}$.)

Given the functions $x_i = x_i(u_1, u_2, u_3)$ relating the two coordinate systems, we want a general expression for the unit vectors $\hat{e}_i$ and their derivatives. Once again we begin with the chain rule,

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u_1}\, du_1 + \frac{\partial\vec{r}}{\partial u_2}\, du_2 + \frac{\partial\vec{r}}{\partial u_3}\, du_3, \tag{2.26}$$

where, as before, the position vector $\vec{r} \equiv (x_1, x_2, x_3)$. Now the vector $\partial\vec{r}/\partial u_i$ is tangent to the u_i coordinate axis (or perhaps more accurately, the u_i coordinate curve), and so defines a coordinate direction $\hat{e}_i$. These unit vectors can be decomposed on the cartesian basis as

$$\hat{e}_i \equiv \frac{1}{h_i}\frac{\partial\vec{r}}{\partial u_i} = \frac{1}{h_i}\left(\hat{\imath}\,\frac{\partial x_1}{\partial u_i} + \hat{\jmath}\,\frac{\partial x_2}{\partial u_i} + \hat{k}\,\frac{\partial x_3}{\partial u_i}\right), \tag{2.27a}$$

where the scale factors h_i are determined by the condition $\hat{e}_i \cdot \hat{e}_i = 1$,

$$h_i \equiv \left|\frac{\partial\vec{r}}{\partial u_i}\right| = \sqrt{\left(\frac{\partial x_1}{\partial u_i}\right)^2 + \left(\frac{\partial x_2}{\partial u_i}\right)^2 + \left(\frac{\partial x_3}{\partial u_i}\right)^2}. \tag{2.27b}$$

In general, the $\hat{e}_i$ and the h_i are both functions of position $\vec{r}$ — that's how we got into this mess in the first place. Instead of staying fixed, the triplet of unit vectors rotates as we move around in space.

Note that a scale factor is given by the ratio of position to curvilinear coordinate. So even though the coordinates u_i may not have units of distance, the scale factors guarantee that the displacement does,

$$d\vec{r} = \hat{e}_1\, h_1 du_1 + \hat{e}_2\, h_2 du_2 + \hat{e}_3\, h_3 du_3. \tag{2.28}$$

We saw this for cylindrical and spherical coordinates in (2.22) and (2.25).

The scale factors are relevant not just for calculations of distance. Suppose we want to find the area of some region S in $\mathbb{R}^2$. In cartesian coordinates this is conceptually straightforward: divide the region into little rectangles with sides Δx, Δy and add up the area of these rectangles. The more rectangles we cram into the region, the better our sum approximates the region's true area; in the limit in which the number of these rectangles goes to infinity we get the exact area of S. Of course, in this limit the sides of the rectangles become infinitesimally small and the sum becomes an integral, $\sum_S \Delta x \Delta y \longrightarrow \int_S dx\, dy$.

This much is simple. But what if we choose to calculate the area in, say, polar coordinates? One might naively try the same approach and assert $\int_S dx\, dy = \int_S d\rho\, d\phi$ — in other words, that the infinitesimal measure $dx\, dy$ is equivalent to $d\rho\, d\phi$. This, however, cannot be correct, if only because the units of these two expressions don't match. But a glance at (2.22) shows what's going on. If we cut up S into little rectangles along curves of constant ρ and ϕ (Figure 2.2), the area of each little rectangle is not $\Delta\rho\Delta\phi$ but rather $\Delta\rho \cdot \rho\Delta\phi$; the extra factor is the product $h_\rho h_\phi$. Thus the area element $dx\, dy$ is actually transformed into the familiar $\rho\, d\rho\, d\phi$. Though area on the surface of a sphere deforms differently than inside a circle, the same basic thing occurs in spherical coordinates; in this case, we'd find $h_\theta h_\phi = r^2 \sin\theta$. Similarly, the volume element requires the product of all three h's — so in spherical coordinates, $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$. As before, the extra factors ensure the correct units for area and volume.

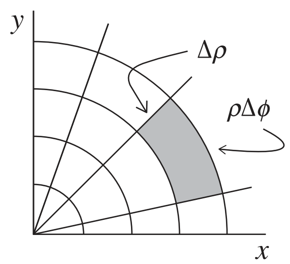


Figure 2.2 Polar scale factors.

Example 2.4 Solid Angle

Consider a circle. A given angle θ subtends an arc length s which grows linearly with the radius r of the circle. Thus the angle can be fully specified by the ratio $\theta = s/r$. This in fact is the definition of a radian; as a ratio of distances it is dimensionless. Since the entire circle has arc length (circumference) $2\pi r$, there are 2π radians in a complete circle.

Now consider a sphere. A given *solid angle* Ω subtends an area A ("two-dimensional arc") which grows *quadratically* with the radius of the sphere. Thus the solid angle can be fully specified by the ratio $\Omega = A/r^2$ (Figure 2.3). This in fact is the definition of a square radian, or *steradian*; as a ratio of areas it is dimensionless. Since the entire sphere has surface area $4\pi r^2$, there are 4π steradians in a complete sphere.[3] Note that in both cases, the radial line is perpendicular to the circular arc or the spherical surface.

An infinitesimal solid angle $d\Omega = dA/r^2$ can be expressed in terms of the scale factors h_θ and h_ϕ,

$$d\Omega = dA/r^2 = h_\theta h_\phi\, d\theta\, d\phi/r^2 = \sin\theta\, d\theta\, d\phi. \tag{2.29}$$

[3] 4π steradians $\approx 41,253$ degrees^2.