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CENTRAL SIMPLE ALGEBRAS AND GALOIS COHOMOLOGY

The first comprehensive, modern introduction to the theory of central simple algebras over arbitrary fields, this book starts from the basics and reaches such advanced results as the Merkurjev–Suslin theorem, a culmination of work initiated by Brauer, Noether, Hasse and Albert and the starting point of current research in motivic cohomology theory by Voevodsky, Suslin, Rost and others.

Assuming only a solid background in algebra, the text covers the basic theory of central simple algebras, methods of Galois descent and Galois cohomology, Severi–Brauer varieties, and techniques in Milnor K-theory and K-cohomology, leading to a full proof of the Merkurjev–Suslin theorem and its application to the characterization of reduced norms. The final chapter rounds off the theory by presenting the results in positive characteristic, including the theorems of Bloch–Gabber–Kato and Izhboldin.

This second edition has been carefully revised and updated, and contains important additional topics.

Philippe Gille is a Research Director (CNRS) at Camille Jordan Institute, Lyon, France. He has written numerous research papers on linear algebraic groups and related structures.

Tamás Szamuely is a Research Adviser at the Alfréd Rényi Institute of Mathematics of the Hungarian Academy of Sciences and a Professor at Central European University, Budapest, Hungary. He is the author of Galois Groups and Fundamental Groups, also published in this series, as well as numerous research papers.

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Central Simple Algebras and Galois Cohomology

Second Edition

PHILIPPE GILLE
Centre National de la Recherche Scientifique (CNRS), Institut Camille Jordan, Lyon

TAMÁS SZAMUELY
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest

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Preface

This book provides a comprehensive and up-to-date introduction to the theory of central simple algebras over arbitrary fields, emphasizing methods of Galois cohomology and (mostly elementary) algebraic geometry. The central result is the Merkurjev–Suslin theorem. As we see it today, this fundamental theorem is at the same time the culmination of the theory of Brauer groups of fields initiated by Brauer, Noether, Hasse and Albert in the 1930s, and a starting point of motivic cohomology theory, a domain which is at the forefront of current research in algebraic geometry and K-theory – suffice it here to mention the recent spectacular results of Voevodsky, Suslin, Rost and others. As a gentle ascent towards the Merkurjev–Suslin theorem, we cover the basic theory of central simple algebras, methods of Galois descent and Galois cohomology, Severi–Brauer varieties, residue maps and, finally, Milnor K-theory and K-cohomology. These chapters also contain a number of noteworthy additional topics. The last chapter of the book rounds off the theory by presenting the results in positive characteristic. For an overview of the contents of each chapter we refer to their introductory sections.

Prerequisites. The book should be accessible to a graduate student or a nonspecialist reader with a solid training in algebra including Galois theory and basic commutative algebra, but no homological algebra. Some familiarity with algebraic geometry is also helpful. Most of the text can be read with a basic knowledge corresponding to, say, the first volume of Shafarevich's text. To help the novice, we summarize in an appendix the results from algebraic geometry we need. The first three sections of Chapter 8 require some familiarity with schemes, and in the proof of one technical statement we are forced to use techniques from Quillen K-theory. However, these may be skipped in a first reading by those willing to accept some 'black boxes'.

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Parts of the book formed the basis of a graduate course by the first author at Université de Paris-Sud and of a lecture series by the two authors at the Alfréd Rényi Institute. We thank both audiences for their pertinent questions and comments, and in particular Endre Szabó who shared his geometric insight with us. Most of the book was written while the first author visited the Rényi Institute in Budapest with a Marie Curie Intra-European Fellowship. The support of the Commission and the hospitality of the Institute are gratefully acknowledged. Last but not least, we are indebted to Diana Gillooly for assuring us a smooth and competent publishing procedure.

Note on the second edition

The first edition of our book has been well received by the mathematical community, and we have received a lot of feedback from experts and graduate students alike. Based partly on their comments, we have tried to correct in this new edition all misprints and inaccuracies known to us. We have updated the text in order to take into account the most important developments during the last ten years, and have also included new material. The most substantial changes to the text of the first edition are as follows.

- We have considerably expanded the material in Section 2.6 on reduced norms.
- There is a new Section 2.8 on a different approach to period-index questions from that of the first edition (which remains in Chapter 4), mainly based on recent work of Antieau and Williams.
- There is a new Section 2.9 on central simple algebras over complete discretely valued fields. Compared to the previous section, this one is much more traditional.

- Section 6.3 has been thoroughly revised: it now includes statements over arbitrary complete discretely valued fields.
- Section 8.6 now includes a theorem of Merkurjev on the generators of the p-torsion in the Brauer group of fields of characteristic different from p.
- Arguably the most important addition to the text is contained in three new sections (8.7–8.9) devoted to the cohomological characterization of reduced norms. This is also a major result in the groundbreaking paper of Merkurjev and Suslin. More recently, it has played a key role in the study of cohomological invariants of algebraic groups. Our approach partly differs from the original one.
- Section 9.1 now includes a recent result of M. Florence on the symbol length in positive characteristic.
- The discussion of the differential symbol in Section 9.5 has been extended to cover mod p^i differential symbols with values in logarithmic de Rham–Witt groups as well.
- There is a new Section 9.8 devoted to Izhboldin's theorem on p-torsion in Milnor K-theory of fields of characteristic p, a fundamental result that was only briefly mentioned in the first edition.

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