1

Probability Distributions and Insurance Applications

1.1 Introduction

This book is about risk theory, with particular emphasis on the two major topics in the field, namely risk models and ruin theory. Risk theory provides a mathematical basis for the study of general insurance risks, and so it is appropriate to start with a brief description of the nature of general insurance risks. The term general insurance essentially applies to an insurance risk that is not a life insurance or health insurance risk, and so the term covers familiar forms of personal insurance such as motor vehicle insurance, home and contents insurance, and travel insurance.

Let us focus on how a motor vehicle insurance policy typically operates from an insurer's point of view. Under such a policy, the insured party pays an amount of money (the premium) to the insurer at the start of the period of insurance cover, which we assume to be one year. The insured party will make a claim under the insurance policy each time the insured party has an accident during the year which results in damage to the motor vehicle, and hence requires repair costs. There are two sources of uncertainty for the insurer: how many claims will the insured party make, and, if claims are made, what will the amounts of these claims be? Thus, if the insurer were to build a probabilistic model to represent its claims outgo under the policy, the model would require a component that modelled the number of claims and another that modelled the amounts of these claims. This is a general framework that applies to modelling claims outgo under any general insurance policy, not just motor vehicle insurance, and we will describe it in greater detail in later chapters.

In this chapter we start with a review of distributions, most of which are commonly used to model either the number of claims arising from an insurance risk or the amounts of individual claims. We then describe mixed distributions before introducing two simple forms of reinsurance arrangement and describing these in mathematical terms. We close the chapter by considering a problem that is important in the context of risk models, namely finding the distribution of a sum of independent and identically distributed random variables.

1.2 Important Discrete Distributions

1.2.1 The Poisson Distribution

When a random variable N has a Poisson distribution with parameter $\lambda > 0$, its probability function is given by

$$\Pr(N = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

for $x = 0, 1, 2, \ldots$. The moment generating function is

$$M_N(t) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = \exp\{\lambda(e^t - 1)\} \tag{1.1}$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^x e^{-\lambda}\frac{\lambda^x}{x!} = \exp\{\lambda(r-1)\}.$$

The moments of N can be found from the moment generating function. For example,

$$M_N'(t) = \lambda e^t M_N(t)$$

and

$$M_N''(t) = \lambda e^t M_N(t) + (\lambda e^t)^2 M_N(t)$$

from which it follows that $E[N] = \lambda$ and $E\left[N^2\right] = \lambda + \lambda^2$, so that $V[N] = \lambda$.

We use the notation $P(\lambda)$ to denote a Poisson distribution with parameter λ.

1.2.2 The Binomial Distribution

When a random variable N has a binomial distribution with parameters n and q, where n is a positive integer and $0 < q < 1$, its probability function is given by

$$\Pr(N = x) = \binom{n}{x} q^x (1-q)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$. The moment generating function is

$$M_N(t) = \sum_{x=0}^{n} e^{tx}\binom{n}{x} q^x (1-q)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (qe^t)^x (1-q)^{n-x}$$
$$= \left(qe^t + 1 - q\right)^n ,$$

and the probability generating function is

$$P_N(r) = (qr + 1 - q)^n .$$

As

$$M'_N(t) = n \left(qe^t + 1 - q\right)^{n-1} qe^t$$

and

$$M''_N(t) = n(n-1) \left(qe^t + 1 - q\right)^{n-2} \left(qe^t\right)^2 + n \left(qe^t + 1 - q\right)^{n-1} qe^t,$$

it follows that $E[N] = nq$, $E\left[N^2\right] = n(n-1)q^2 + nq$ and $V[N] = nq(1-q)$.

We use the notation $B(n, q)$ to denote a binomial distribution with parameters n and q.

1.2.3 The Negative Binomial Distribution

When a random variable N has a negative binomial distribution with parameters $k > 0$ and p, where $0 < p < 1$, its probability function is given by

$$\Pr(N = x) = \binom{k + x - 1}{x} p^k q^x$$

for $x = 0, 1, 2, \ldots$, where $q = 1 - p$. When k is an integer, calculation of the probability function is straightforward as the probability function can be expressed in terms of factorials. An alternative method of calculating the probability function, regardless of whether k is an integer, is recursively as

$$\Pr(N = x + 1) = \frac{k + x}{x + 1} q \Pr(N = x)$$

for $x = 0, 1, 2, \ldots$, with starting value $\Pr(N = 0) = p^k$.

The moment generating function can be found by making use of the identity

$$\sum_{x=0}^{\infty} \Pr(N = x) = 1. \tag{1.2}$$

From this it follows that

$$\sum_{x=0}^{\infty} \binom{k + x - 1}{x} (1 - qe^t)^k (qe^t)^x = 1$$

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provided that $0 < qe^t < 1$. Hence

$$\begin{aligned} M_N(t) &= \sum_{x=0}^{\infty} e^{tx} \binom{k+x-1}{x} p^k q^x \\ &= \frac{p^k}{(1-qe^t)^k} \sum_{x=0}^{\infty} \binom{k+x-1}{x} (1-qe^t)^k (qe^t)^x \\ &= \left(\frac{p}{1-qe^t} \right)^k \end{aligned}$$

provided that $0 < qe^t < 1$, or, equivalently, $t < -\log q$. Similarly, the probability generating function is

$$P_N(r) = \left(\frac{p}{1-qr} \right)^k .$$

Moments of this distribution can be found by differentiating the moment generating function, and the mean and variance are given by $E[N] = kq/p$ and $V[N] = kq/p^2$.

Equality (1.2) trivially gives

$$\sum_{x=1}^{\infty} \binom{k+x-1}{x} p^k q^x = 1 - p^k, \tag{1.3}$$

a result we shall use in Section 4.5.1.

We use the notation $NB(k, p)$ to denote a negative binomial distribution with parameters k and p.

1.2.4 The Geometric Distribution

The geometric distribution is a special case of the negative binomial distribution. When the negative binomial parameter k is 1, the distribution is called a geometric distribution with parameter p and the probability function is

$$\Pr(N = x) = pq^x$$

for $x = 0, 1, 2, \ldots$. From above, it follows that $E[N] = q/p$, $V[N] = q/p^2$ and

$$M_N(t) = \frac{p}{1-qe^t}$$

for $t < -\log q$.

This distribution plays an important role in ruin theory, as will be seen in Chapter 7.

1.3 Important Continuous Distributions

1.3.1 The Gamma Distribution

When a random variable X has a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, its density function is given by

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

for $x > 0$, where $\Gamma(\alpha)$ is the gamma function, defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

In the special case when α is an integer the distribution is also known as an Erlang distribution, and repeated integration by parts gives the distribution function as

$$F(x) = 1 - \sum_{j=0}^{\alpha-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!}$$

for $x \geq 0$. The moments and moment generating function of the gamma distribution can be found by noting that

$$\int_0^\infty f(x) dx = 1$$

yields

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}. \tag{1.4}$$

The nth moment is

$$E\left[X^n\right] = \int_0^\infty x^n \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{n+\alpha-1} e^{-\lambda x} dx,$$

and from identity (1.4) it follows that

$$E\left[X^n\right] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\lambda^n}. \tag{1.5}$$

In particular, $E[X] = \alpha/\lambda$ and $E\left[X^2\right] = \alpha(\alpha+1)/\lambda^2$, so that $V[X] = \alpha/\lambda^2$.

We can find the moment generating function in a similar fashion. As

$$M_X(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx\,, \tag{1.6}$$

application of identity (1.4) gives

$$M_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda - t)^\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^\alpha. \tag{1.7}$$

Note that in identity (1.4), $\lambda > 0$. Hence, in order to apply (1.4) to (1.6) we require that $\lambda - t > 0$, so that the moment generating function exists when $t < \lambda$.

A result that will be used in Section 4.8.2 is that the coefficient of skewness of X, which we denote by $Sk[X]$, is $2/\sqrt{\alpha}$. This follows from the definition of the coefficient of skewness, namely third central moment divided by standard deviation cubed, and the fact that the third central moment is

$$\begin{aligned} E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] &= E\left[X^3\right] - 3\frac{\alpha}{\lambda}E[X^2] + 2\left(\frac{\alpha}{\lambda}\right)^3 \\ &= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\alpha^3}{\lambda^3} \\ &= \frac{2\alpha}{\lambda^3}. \end{aligned}$$

We use the notation $\gamma(\alpha, \lambda)$ to denote a gamma distribution with parameters α and λ.

1.3.2 The Exponential Distribution

The exponential distribution is a special case of the gamma distribution. It is just a gamma distribution with parameter $\alpha = 1$. Hence, the exponential distribution with parameter $\lambda > 0$ has density function

$$f(x) = \lambda e^{-\lambda x}$$

for $x > 0$, and has distribution function

$$F(x) = 1 - e^{-\lambda x}$$

for $x \geq 0$. From equation (1.5), the nth moment of the distribution is

$$E\left[X^n\right] = \frac{n!}{\lambda^n},$$

and from equation (1.7) the moment generating function is

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

for $t < \lambda$.

1.3.3 The Pareto Distribution

When a random variable X has a Pareto distribution with parameters $\alpha > 0$ and $\lambda > 0$, its density function is given by

$$f(x) = \frac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha+1}}$$

for $x > 0$. Integrating this density we find that the distribution function is

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$$

for $x \geq 0$. Whenever moments of the distribution exist, they can be found from

$$E[X^n] = \int_0^{\infty} x^n f(x)dx$$

by integration by parts. However, they can also be found individually using the following approach. Since the integral of the density function over $(0, \infty)$ equals 1, we have

$$\int_0^{\infty} \frac{dx}{(\lambda + x)^{\alpha+1}} = \frac{1}{\alpha\lambda^{\alpha}},$$

an identity which holds provided that $\alpha > 0$. To find $E[X]$, we can write

$$E[X] = \int_0^{\infty} xf(x)dx = \int_0^{\infty} (x + \lambda - \lambda)f(x)dx = \int_0^{\infty} (x + \lambda)f(x)dx - \lambda,$$

and inserting for f we have

$$E[X] = \int_0^{\infty} \frac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha}}dx - \lambda.$$

We can evaluate the integral expression by rewriting the integrand in terms of a Pareto density function with parameters $\alpha - 1$ and λ. Thus,

$$E[X] = \frac{\alpha\lambda}{\alpha - 1}\int_0^{\infty} \frac{(\alpha - 1)\lambda^{\alpha-1}}{(\lambda + x)^{\alpha}}dx - \lambda, \tag{1.8}$$

and since the integral equals 1,

$$E[X] = \frac{\alpha\lambda}{\alpha - 1} - \lambda = \frac{\lambda}{\alpha - 1}.$$

It is important to note that the integrand in equation (1.8) is a Pareto density function only if $\alpha > 1$, and hence $E[X]$ exists only for $\alpha > 1$. Similarly, we can find $E\left[X^2\right]$ from

$$E\left[X^2\right] = \int_0^\infty \left((x+\lambda)^2 - 2\lambda x - \lambda^2\right) f(x)dx$$
$$= \int_0^\infty (x+\lambda)^2 f(x)dx - 2\lambda E[X] - \lambda^2.$$

Proceeding as in the case of $E[X]$ we can show that

$$E\left[X^2\right] = \frac{2\lambda^2}{(\alpha-1)(\alpha-2)}$$

provided that $\alpha > 2$, and hence that

$$V[X] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}.$$

An alternative method of finding moments of the Pareto distribution is given in Exercise 5 at the end of this chapter.

We use the notation $Pa(\alpha, \lambda)$ to denote a Pareto distribution with parameters α and λ.

1.3.4 The Normal Distribution

When a random variable X has a normal distribution with parameters μ and σ^2, its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

for $-\infty < x < \infty$. We use the notation $N(\mu, \sigma^2)$ to denote a normal distribution with parameters μ and σ^2.

The standard normal distribution has parameters 0 and 1 and its distribution function is denoted Φ, where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-z^2/2\right\} dz.$$

A key relationship is that if $X \sim N(\mu, \sigma^2)$ and if $Z = (X-\mu)/\sigma$, then $Z \sim N(0,1)$.

The moment generating function is

$$M_X(t) = \exp\left\{\mu t + \tfrac{1}{2}\sigma^2 t^2\right\} \tag{1.9}$$

from which it can be shown (see Exercise 7) that $E[X] = \mu$ and $V[X] = \sigma^2$.

1.3.5 The Lognormal Distribution

When a random variable X has a lognormal distribution with parameters μ and σ, where $-\infty < \mu < \infty$ and $\sigma > 0$, its density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}$$

for $x > 0$. The distribution function can be obtained by integrating the density function as follows:

$$F(x) = \int_0^x \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} dy,$$

and the substitution $z = \log y$ yields

$$F(x) = \int_{-\infty}^{\log x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(z - \mu)^2}{2\sigma^2}\right\} dz.$$

As the integrand is the $N(\mu, \sigma^2)$ density function,

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right).$$

Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function.

We use the notation $LN(\mu, \sigma)$ to denote a lognormal distribution with parameters μ and σ. From the preceding argument it follows that if $X \sim LN(\mu, \sigma)$, then $\log X \sim N(\mu, \sigma^2)$.

This relationship between normal and lognormal distributions is extremely useful, particularly in deriving moments. If $X \sim LN(\mu, \sigma)$ and $Y = \log X$, then

$$E\left[X^n\right] = E\left[e^{nY}\right] = M_Y(n) = \exp\left\{\mu n + \tfrac{1}{2}\sigma^2 n^2\right\},$$

where the final equality follows by equation (1.9).

1.4 Mixed Distributions

Many of the distributions encountered in this book are mixed distributions. To illustrate the idea of a mixed distribution, let X be exponentially distributed with mean 100, and let the random variable Y be defined by

$$Y = \begin{cases} 0 & \text{if } X < 20 \\ X - 20 & \text{if } 20 \le X < 300 \\ 280 & \text{if } X \ge 300 \end{cases}.$$

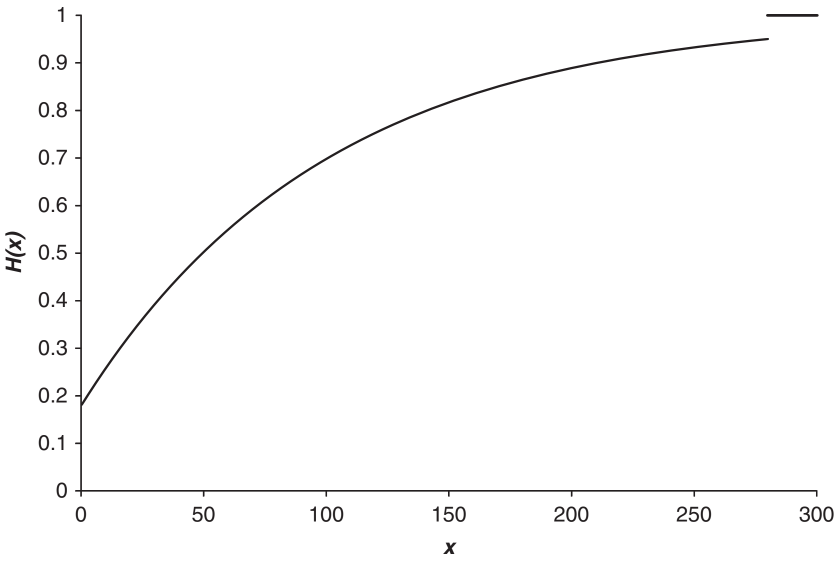

Figure 1.1 The distribution function H.

Then

$$\Pr(Y = 0) = \Pr(X < 20) = 1 - e^{-0.2} = 0.1813,$$

and similarly $\Pr(Y = 280) = 0.0498$. Thus, Y has masses of probability at the points 0 and 280. However, in the interval $(0, 280)$, the distribution of Y is continuous, with, for example,

$$\Pr(30 < Y \le 100) = \Pr(50 < X \le 120) = 0.3053.$$

Figure 1.1 shows the distribution function, H, of Y. Note that there are jumps at 0 and 280, corresponding to the masses of probability at these points. As the distribution function is differentiable in the interval $(0, 280)$, Y has a density function in this interval. Letting h denote the density function of Y, the moments of Y can be found from

$$E\left[Y^r\right] = \int_0^{280} x^r h(x)dx + 280^r \Pr(Y = 280).$$

At certain points in this book, it will be convenient to use Stieltjes integral notation, so that we do not have to specify whether a distribution is discrete, continuous or mixed. In this notation, we write the rth moment of Y as

$$E\left[Y^r\right] = \int_0^{\infty} x^r dH(x).$$