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Introduction

The main objective of this book is to study the structured dependence between stochastic processes, such as for example Markov chains, conditional Markov chains, and special semimartingales, as well as between Markov families. In particular, we devote considerable attention to the modeling of structured dependence.

In a nutshell, the structured dependence between stochastic processes is the stochastic dependence between these processes subject to the relevant structural requirements.

Some major practical motivations for modeling the structured dependence between stochastic processes can be briefly summarized as follows. This methodology allows one to build models for multivariate stochastic differential systems subject to the constraint that the idiosyncratic dynamical properties of the multivariate system, that is, the dynamics of the univariate constituents of the modeled multivariate system, are matched with the univariate data. Both the multivariate and the univariate dynamics can be modeled as desired. For example, in case of Markov structures, both the multivariate and the univariate dynamics are Markovian, which allows one to tap in to the rich theory and practice of Markov processes. Moreover, and quite importantly, by using structured-dependence modeling one can separate estimation of the univariate, idiosyncratic, parameters of the model from estimation of the parameters accounting for the stochastic dependence *between* the univariate constituents of the model. In particular, within the structured-dependence model one can match the dynamics of the univariate constituents of the system that are estimated from the univariate data.

Let $\mathcal{P}$ represent a certain property of stochastic processes, such as the Feller–Markov property or the special semimartingale property. Let us fix an integer $n > 1$ and let $\mathcal{P}^n$ denote a class of n-variate stochastic processes[1] that take values in $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ and that feature the property $\mathcal{P}$. The problem of modeling the structured dependence between stochastic processes can be summarized as follows.

Given a collection of univariate stochastic processes, say $Y^i,\ i = 1, \ldots, n$, taking values in $\mathcal{X}_i,\ \ i = 1, \ldots, n$, respectively, and all featuring property $\mathcal{P}$, construct

[1] We refer to Appendix A for the key definitions in the area of stochastic analysis that are used throughout this book.

n-variate stochastic processes, say $X = (X^1, \ldots, X^n)$, such that $X \in \mathcal{P}^n$ and the structural properties of X^i are the same as the structural properties of Y^i for all $i = 1, \ldots, n$. For example, if $\mathcal{P}$ represents the Markov property and the structural property of process Y^i is that it is a Markov chain with generator function $\Lambda^i(\cdot)$, $i = 1, \ldots, n$, then we want X to be an n-variate Markov chain and X^i to be a Markov chain with generator function $\Lambda^i(\cdot)$, $i = 1, \ldots, n$. Or, if $\mathcal{P}$ represents the special semimartingale property and the structural property of process Y^i is that it is a special semimartingale process with characteristic triple (B^i, C^i, ν^i), $i = 1, \ldots, n$, then we want X to be a special semimartingale and X^i to be a special semimartingale process with characteristic triple (B^i, C^i, ν^i). Any such process X is called an n-dimensional *$\mathcal{P}$-structure* for Y^i, $i = 1, \ldots, n$, and the processes Y^i, $i = 1, \ldots, n$, are called the *predetermined margins for* X. We will sometimes use the term *stochastic structure* if reference to a specific class $\mathcal{P}$ is not needed.

In many cases of interest the structural properties of a stochastic process can be used to determine the law of the process. For example, in case of a Markov process starting at a given time, the distribution of the process at this time and the generator of the process determine the law of this process. Also, with semimartingales, it is frequently the case, albeit not always, that the initial law of a semimartingale and its characteristic triple determine its law. In such situations, any $\mathcal{P}$-structure X for Y^i, $i = 1, \ldots, n$, provides the additional benefit that the law of X^i may be chosen so that it agrees with the law of Y^i, $i = 1, \ldots, n$. If this can be done then we refer to $\mathcal{P}$-structure X as a *$\mathcal{P}$-copula structure*.

In this book we focus on constructing $\mathcal{P}$-structures, which naturally lead to $\mathcal{P}$-copula structures.

The problem of constructing a $\mathcal{P}$-copula structure clearly is reminiscent of one of the classical problems in probability: given a family of $\mathbb{R}^1$-valued random variables, say U^i, $i = 1, \ldots, n$, construct $\mathbb{R}^n$-valued random variables, say $V = (V^1, \ldots, V^n)$, such that the law of V^i is the same as the law of U^i for all $i = 1, \ldots, n$. The elegant and satisfactory solution to this problem is given in terms of copulae (see Appendix A) and the celebrated Sklar theorem (see Sklar (1959)). Here is a statement of the Sklar theorem,

Theorem 1.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, and let $V = (V^1, \ldots, V^n)$ be an $\mathbb{R}^n$-valued random variable on this space. Next, let*

$$F(v^1, \ldots, v^n) = \mathbb{P}(V^1 \leq v^1, \ldots, V^n \leq v^n)$$

be the cumulative probability distribution function of V, and let $F^i(v^i) = \mathbb{P}(V^i \leq v^i)$, $i = 1, \ldots, n$ be the corresponding marginal cumulative probability distributions. Then there exists a function C, called the copula, such that

$$F(v^1, \ldots, v^n) = C\left(F^1(v^1), \ldots, F^n(v^n)\right)$$

for each $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$. Moreover, if the marginal distributions are continuous then the copula C is unique.

In general, the Sklar theorem does not extend to the case of (infinite-dimensional) vector-space-valued random variables such as are found in a stochastic process.[2] This means that, in general, there does not exist a copula that could be used to construct a $\mathcal{P}$-copula structure for $Y^i,\ i = 1,\dots,n$. Thus, other methods need to be used for this purpose. A possible method, based on the theory developed in this book, would be to construct a $\mathcal{P}$-structure for $Y^i,\ i = 1,\dots,n$, which, in cases when structural properties of the process can be used to determine its law, would lead to constructing a $\mathcal{P}$-copula structure for $Y^i,\ i = 1,\dots,n$.

In this volume we propose methods of constructing $\mathcal{P}$-structures for various classes of stochastic processes: Feller–Markov processes, Markov chains, conditional Markov chains, special semimartingales, Archimedean survival processes, and generalized Hawkes processes.

The practical importance of stochastic structures cannot be exaggerated. In fact, stochastic structures provide great flexibility for modeling the dependence between dynamic random phenomena with the preservation of important structural features and marginal features. Moreover, these structures allow for separation of the estimation of the marginal structural properties of a multivariate dynamical system from the estimation of the dependence structure between the margins. This feature is key for the efficient implementation of stochastic structures.

It needs to be remarked that in the case $n = 2$ the concept of a stochastic structure is related to the concept of the coupling between two probability measures. In fact, if $\mathcal{X}^1 = \mathcal{X}^2$ then the law of a bivariate stochastic structure $X = (X^1, X^2)$, considered as a probability measure on the canonical space supported by $\mathcal{X}^1 \times \mathcal{X}^2$, is a coupling between the laws of X^1 and X^2.

The theory of structured dependence is closely related to the theory of consistency for multivariate stochastic processes. The theory of consistency studies the following question: which processes $X = (X^1,\dots,X^n) \in \mathcal{P}^n$ are such that all X^i, $i = 1,\dots,n$, feature property $\mathcal{P}$? We devote much attention in this book to consistency.

In the case of Markov processes, the study of Markovian consistency is related to the study of so-called Markov functions. The latter considers the following issue: let M be a Markov process taking values in $\mathcal{M}$, and let $\phi : \mathcal{M} \to \mathcal{Y}$. Then the question is, for which processes M and for which functions ϕ is the process Y, given as $Y_t = \phi(M_t)$, a Markov process? In the case of Markovian consistency, one asks this question for $M = X = (X^1,\dots,X^n)$ and for functions ϕ_i given as the coordinate projections, that is, $\phi_i(x^1,\dots,x^n) = x^i$. Markov functions were studied in Chapter X, Section 6, in Dynkin (1965), Rogers and Pitman (1981), and Kurtz (1998), among others. In this book we aim to contribute significantly to the theory and practice of Markov functions by providing an in-depth study of Markovian invariance under coordinate projection.

We tackle only the consistency of a process $X = (X^1,\dots,X^n)$ with respect to its individual coordinates $X^i,\ i = 1,\dots,n$. Likewise, we study only stochastic

[2] We refer to Scarsini (1989) and to Bielecki et al. (2008b) for relevant discussions.

structures subject to univariate marginal constraints. In more generality, one would study these two aspects of structural dependence with regard to groups of coordinates: $X^{\mathcal{I}_m} = (X^i,\ i \in \mathcal{I}_m)$, $m = 1,\dots,M$, where $\{\mathcal{I}_m : m = 1,\dots,M\}$ is a partition of the set $\{1,\dots,n\}$.

The book is organized as follows. In Part One we provide a study of consistency for multivariate Markov families, multivariate Markov chains, multivariate conditional Markov chains, and multivariate special semimartingales. This study underlies the study of stochastic structures conducted in Part Two. Part Three is devoted to the investigation of issues related to consistency and structured dependence in the case of Archimedean survival processes and generalized multivariate Hawkes processes. In particular, we introduce and study generalized multivariate Hawkes processes, as well as the related concept of Hawkes structure. This concept turns out to be important and useful in numerous applications. Part Four illustrates the theory of structured dependence with numerous examples of the practical implementation of stochastic structures. Finally, we provide some relevant technical background in the Appendices.