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# MODERN ANALYSIS OF AUTOMORPHIC FORMS BY EXAMPLE, VOLUME 1

This is Volume 1 of a two-volume book that provides a self-contained introduction to the theory and application of automorphic forms, using examples to illustrate several critical analytical concepts surrounding and supporting the theory of automorphic forms. The two-volume book treats three instances, starting with some small unimodular examples, followed by adelic GL2, and finally GLn. Volume 1 features critical results, which are proven carefully and in detail, including discrete decomposition of cuspforms, meromorphic continuation of Eisenstein series, spectral decomposition of pseudo-Eisenstein series, and automorphic Plancherel theorem. Volume 2 features automorphic Green's functions, metrics and topologies on natural function spaces, unbounded operators, vector-valued integrals, vector-valued holomorphic functions, and asymptotics. With numerous proofs and extensive examples, this classroom-tested introductory text is meant for a second-year or advanced graduate course in automorphic forms, and also as a resource for researchers working in automorphic forms, analytic number theory, and related fields.

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# Modern Analysis of Automorphic Forms by Example, Volume 1

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# Contents

# Introduction and Historical Notes

The aim of this book is to offer persuasive proof of several important analytical results about automorphic forms, among them spectral decompositions of spaces of automorphic forms, discrete decompositions of spaces of cuspforms, meromorphic continuation of Eisenstein series, spectral synthesis of automorphic forms, a Plancherel theorem, and various notions of convergence of spectral expansions. Rather than assuming prior knowledge of the necessary analysis or giving extensive external references, this text provides customized discussions of that background, especially of ideas from 20th-century analysis that are often neglected in the contemporary standard curriculum. Similarly, I avoid assumptions of background that would certainly be useful in studying automorphic forms but that beginners cannot be expected to have. Therefore, I have kept external references to a minimum, treating the modern analysis and other background as a significant part of the discussion.

Not only for reasons of space, the treatment of automorphic forms is deliberately neither systematic nor complete, but instead provides three families of examples, in all cases aiming to illustrate aspects beyond the introductory case of $SL_2(\mathbb{Z})$ and its congruence subgroups.

The first three chapters set up the three families of examples, proving essential preparatory results and many of the basic facts about automorphic forms, while merely stating results whose proofs are more sophisticated or difficult. The proofs of the more difficult results occupy the remainder of the book, as in many cases the arguments require various ideas not visible in the statements.

The first family of examples is introduced in Chapter 1, consisting of *waveforms* on quotients having dimensions $2, 3, 4, 5$ with a single *cusp*, which is just a *point*. In the two-dimensional case, the space on which the functions live is the usual quotient $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ of the complex upper half-plane $\mathfrak{H}$. The three-dimensional case is related to $SL_2(\mathbb{Z}[i])$, and the four-dimensional and

five-dimensional cases are similarly explicitly described. Basic discussion of the physical spaces themselves involves explication of the groups acting on them, and decompositions of these groups in terms of subgroups, as well as the expression of the physical spaces as $G/K$ for $K$ a maximal compact subgroup of $G$. There are natural invariant measures and integrals on $G/K$ and on $\Gamma\backslash G/K$ whose salient properties can be described quickly, with proofs deferred to a later point. Similarly, a natural Laplace-Beltrami operator $\Delta$ on $G/K$ and $\Gamma\backslash G/K$ can be described easily, but with proofs deferred. The first serious result specific to automorphic forms is about *reduction theory*, that is, determination of a nice set in $G/K$ that *surjects* to the quotient $\Gamma\backslash G/K$, for specific *discrete* subgroups $\Gamma$ of $G$. The four examples in this simplest scenario all admit very simple sets of representatives, called *Siegel sets*, in every case a product of a ray and a box, with Fourier expansions possible along the box-coordinate, consonant with a decomposition of part of the group $G$ (Iwasawa decomposition). This greatly simplifies both statements and proofs of fundamental theorems.

In the simplest family of examples, the space of *cuspforms* consists of those functions on the quotient $\Gamma\backslash G/K$ with $0^{\text{th}}$ Fourier coefficient identically 0. The basic theorem, quite nontrivial to prove, is that the space of cuspforms in $L^2(\Gamma\backslash G/K)$ has a basis consisting of eigenfunctions for the invariant Laplacian $\Delta$. This result is one form of the *discrete decomposition of cuspforms*. We delay its proof, which uses many ideas not apparent in the statement of the theorem. The orthogonal complement to cuspforms in $L^2(\Gamma\backslash G/K)$ is readily characterized as the space of *pseudo-Eisenstein series*, parametrized here by test functions on $(0, +\infty)$. However, these simple, explicit automorphic forms are never eigenfunctions for $\Delta$. Rather, via Euclidean Fourier-Mellin inversion, they are expressible as *integrals* of (genuine) Eisenstein series, the latter eigenfunctions for $\Delta$, but unfortunately not in $L^2(\Gamma\backslash G/K)$. Further, it turns out that the best expression of pseudo-Eisenstein series in terms of genuine Eisenstein series $E_s$ involves the latter with complex parameter outside the region of convergence of the defining series. Thus arises the need to *meromorphically continue* the Eisenstein series in that complex parameter. Genuine proof of meromorphic continuation, with control over the behavior of the meromorphically continued function, is another basic but nontrivial result, whose proof is delayed. Granting those postponed proofs, a *Plancherel theorem* for the space of pseudo-Eisenstein series follows from their expansion in terms of genuine Eisenstein series, together with attention to integrals as *vector-valued* (rather than merely numerical), with the important corollary that such integrals commute with continuous operators on the vector space. This and other aspects of vector-valued integrals are treated at length in an appendix. Then we obtain the Plancherel theorem for the whole space of $L^2$ waveforms. Even for the simplest

examples, these few issues illustrate the goals of this book: discrete decomposition of spaces of cuspforms, meromorphic continuation of Eisenstein series, and a Plancherel theorem.

In Chapter 2 is the second family of examples, *adele groups* $GL_2$ over number fields. These examples subsume classical examples of quotient $\Gamma_0(N)\backslash\mathfrak{H}$ with several cusps, reconstituting things so that operationally there is a *single* cusp. Also, examples of Hilbert modular groups and Hilbert modular forms are subsumed by rewriting things so that the vagaries of class numbers and unit groups become irrelevant. Assuming some basic algebraic number theory, we prove $p$-adic analogues of the group decomposition results proven earlier in Chapter 1 for the purely archimedean examples. Integral operators made from $C_c^o$ functions on the $p$-adic factor groups, known as *Hecke operators*, are reasonable $p$-adic analogues of the archimedean factors' $\Delta$, although the same integral operators do make the same sense on archimedean factors. Again, the first serious result for these examples is that of *reduction theory*, namely, that there is a single nice set, an adelic form of a *Siegel set*, again nearly the product of a ray and a box, that surjects to the quotient $Z^+GL_2(k)\backslash GL_2(\mathbb{A})$, where $Z^+$ is itself a ray in the center of the group. The first serious analytical result is again about *discrete decomposition of spaces of cuspforms*, where now relevant operators are both the invariant Laplacians and the Hecke operators. Again, the deferred proof is much more substantial than the statement and needs ideas not visible in the assertion itself. The orthogonal complement to cuspforms is again describable as the $L^2$ span of *pseudo-Eisenstein series*, now with a discrete parameter, a Hecke character (grossencharacter) of the ground field, in addition to the test function on $(0, +\infty)$. The pseudo-Eisenstein series are never eigenfunctions for invariant Laplacians or for Hecke operators. Within each family, indexed by Hecke characters, every pseudo-Eisenstein series again decomposes via Euclidean Fourier-Mellin inversion as an integral of (genuine) Eisenstein series with the same discrete parameter. The genuine Eisenstein series are eigenfunctions for invariant Laplacians and are eigenfunctions for Hecke operators at almost all finite places, but are not square-integrable. Again, the best assertion of spectral decomposition requires a meromorphic continuation of the genuine Eisenstein series in the continuous parameter. Then a Plancherel theorem for pseudo-Eisenstein series for each discrete parameter value follows from the integral representation in terms of genuine Eisenstein series and general properties of vector-valued integrals. These are assembled into a Plancherel theorem for all $L^2$ automorphic forms. An appendix computes *periods* of Eisenstein series along copies of $GL_1(\widetilde{k})$ of quadratic field extensions $\widetilde{k}$ of the ground field.

Chapter 3 treats the most complicated of the three families of examples, including automorphic forms for $SL_n(\mathbb{Z})$, both purely archimedean and adelic.

Again, some relatively elementary set-up regarding group decompositions is necessary and carried out immediately. Identification of invariant differential operators and Hecke operators at finite places is generally similar to that for the previous example $GL_2$. A significant change is the proliferation of types of *parabolic subgroups* (essentially, subgroups conjugate to subgroups containing upper-triangular matrices). This somewhat complicates the notion of *cuspform*, although the general idea, that zeroth Fourier coefficients vanish, is still correct, if suitably interpreted. Again, the space of square-integrable cuspforms decomposes discretely, although the complexity of the proof for these examples increases significantly and is again delayed. The increased complication of parabolic subgroups also complicates the description of the orthogonal complement to cuspforms in terms of pseudo-Eisenstein series. For purposes of spectral decomposition, the discrete parameters now become more complicated than the $GL_2$ situation: *cuspforms* on the Levi components (diagonal blocks) in the parabolics generalize the role of Hecke characters. Further, the continuous complex parametrizations need to be over larger-dimensional Euclidean spaces. Thus, I restrict attention to the two extreme cases: minimal parabolics (also called *Borel subgroups*), consisting exactly of upper-triangular matrices, and *maximal proper* parabolics, which have exactly two diagonal blocks. The minimal parabolics use no cuspidal data but for $SL_n(\mathbb{Z})$ have an $(n-1)$-dimensional complex parameter. The maximal proper parabolics have just a one-dimensional complex parameter but typically need two cuspforms on smaller groups, one on each of the two diagonal blocks. The general qualitative result that the $L^2$ orthogonal complement to cuspforms is spanned by pseudo-Eisenstein series of various types does still hold, and the various types of pseudo-Eisenstein series are integrals of genuine Eisenstein series with the same discrete parameters. Again, the best description of these integrals requires the meromorphic continuation of the Eisenstein series. For nonmaximal parabolics, Bochner's lemma (recalled and proven in an appendix) reduces the problem of meromorphic continuation to the maximal proper parabolic case, with cuspidal data on the Levi components. Elementary devices such as Poisson summation, which suffice for meromorphic continuation for $GL_2$, as seen in the appendix to Chapter 2, are inadequate to prove meromorphic continuation involving the nonelementary cuspidal data. I defer the proof. Plancherel theorems for the spectral fragments follow from the integral representations in terms of genuine Eisenstein series, together with properties of vector-valued integrals.

The rest of the book gives proofs of those foundational analytical results, discreteness of cuspforms and meromorphic continuation of Eisenstein series, at various levels of complication and by various devices. Perhaps surprisingly,

the required analytical underpinnings are considerably more substantial than an unsuspecting or innocent bystander might imagine. Further, not everyone interested in the *truth* of foundational analytical facts about automorphic forms will necessarily care about their *proofs*, especially upon discovery that that burden is greater than anticipated. These obvious points reasonably explain the compromises made in many sources. Nevertheless, rather than either gloss over the analytical issues, refer to encyclopedic treatments of modern analysis on a scope quite unnecessary for our immediate interests, or give suggestive but misleading neoclassical heuristics masquerading as adequate arguments for what is truly needed, the remaining bulk of the book aims to discuss analytical issues at a technical level truly sufficient to convert appealing heuristics to persuasive, genuine proofs. For that matter, one's own lack of interest in the proofs might provide all the more interest in knowing that things widely believed are in fact provable by standard methods.

Chapter 4 explains enough Lie theory to understand the invariant differential operators on the ambient archimedean groups $G$, both in the simplest small examples and, more generally, determining the invariant Laplace-Beltrami operators explicitly in coordinates on the four simplest examples.

Chapter 5 explains how to integrate on quotients, without concern for explicit sets of representatives. Although in very simple situations, such as quotients $\mathbb{R}/\mathbb{Z}$ (the circle), it is easy to manipulate sets of representatives (the interval $[0, 1]$ for the circle), this eventually becomes infeasible, despite the traditional example of the explicit fundamental domain for $SL_2(\mathbb{Z})$ acting on the upper half-plane $\mathfrak{H}$. That is, much of the picturesque detail is actually inessential, which is fortunate because that level of detail is also unsustainable in all but the simplest examples.

Chapter 6 introduces natural actions of groups on spaces of functions on *physical* spaces on which the groups act. In some contexts, one might make a more elaborate *representation theory* formalism here, but it is possible to reap many of the benefits of the ideas of representation theory without the usual superstructure. That is, the *idea* of a linear action of a topological group on a topological vector space of functions on a physical space is the beneficial notion, with or without *classification*. It is true that at certain technical moments, classification results are crucial, so although we do not prove either the Borel-Casselman-Matsumoto classification in the $p$-adic case [Borel 1976], [Matsumoto 1977], [Casselman 1980], nor the subrepresentation theorem [Casselman 1978/1980], [Casselman-Miličić 1982] in the archimedean case, ideally the roles of these results are made clear. Classification results per se, although difficult and interesting problems, do not necessarily affect the foundational analytic aspects of automorphic forms.

Chapter 7 proves the discreteness of spaces of cuspforms, in various senses, in examples of varying complexity. Here, it becomes apparent that genuine proofs, as opposed to heuristics, require some sophistication concerning topologies on natural function spaces, beyond the typical Hilbert, Banach, and Fréchet spaces. Here again, there is a forward reference to the extended appendix on function spaces and classes of topological vector spaces necessary for practical analysis. Further, even less immediately apparent, but in fact already needed in the discussion of decomposition of pseudo-Eisenstein series in terms of genuine Eisenstein series, we need a coherent and effective theory of *vector-valued integrals*, a complete, succinct form given in the corresponding appendix, following Gelfand and Pettis, making explicit the most important corollaries on uniqueness of invariant functions, differentiation under integral signs with respect to parameters, and related.

Chapter 8 fills an unobvious need, proving that automorphic forms that are of *moderate growth* and are eigenfunctions, for Laplacians have asymptotics given by their *constant terms*. In the smaller examples, it is easy to make this precise. For $SL_n$ with $n \geq 3$, some effort is required for an accurate statement. As corollaries, $L^2$ cuspforms that are eigenfunctions are of rapid decay, and Eisenstein series have relatively simple asymptotics given by their constant terms. Thus, we discover again the need to prove that Eisenstein series have vector-valued meromorphic continuations, specifically as moderate-growth functions.

Chapter 9 carefully develops ideas concerning unbounded symmetric operators on Hilbert spaces, thinking especially of operators related to Laplacians $\Delta$, and especially those such that $(\Delta - \lambda)^{-1}$ is a compact-operator-valued meromorphic function of $\lambda \in \mathbb{C}$. On one hand, even a naive conception of the general behavior of Laplacians is fairly accurate, but this is due to a subtle fact that needs proof, namely, the *essential self-adjointness* of Laplacians on natural spaces such as $\mathbb{R}^n$, multi-toruses $\mathbb{T}^n$, spaces $G/K$, and even spaces $\Gamma\backslash G/K$. This has a precise sense: the (invariant) Laplacian restricted to *test functions* has a *unique* self-adjoint extension, which then is necessarily its *graph-closure*. Thus, the naive presumption, implicit or explicit, that the graph closure is a (maximal) self-adjoint extension is *correct*. On the other hand, the proof of meromorphic continuation of Eisenstein series in [Colin de Verdière 1981, 1982/1983] makes essential use of some quite counterintuitive features of (Friedrichs's) self-adjoint extensions of *restrictions* of self-adjoint operators, which therefore merit careful attention. In this context, the basic examples are the usual Sobolev spaces on $\mathbb{T}$ or $\mathbb{R}$ and the quantum harmonic oscillator $-\Delta + x^2$ on $\mathbb{R}$. An appendix recalls the proof of the spectral theorem for compact, self-adjoint operators.

Chapter 10 extends the idea from [Lax-Phillips 1976] to prove that larger spaces than spaces of cuspforms decompose *discretely* under the action of self-adjoint extensions $\widetilde{\Delta}_a$ of suitable restrictions $\Delta_a$ of Laplacians. Namely, the space of *pseudo-cuspforms* $L^2_a$ at cutoff height $a$ is specified, not by requiring constant terms to vanish *entirely*, but by requiring that all constant terms vanish above height $a$. The discrete decomposition is proven, as expected, by showing that the resolvent $(\widetilde{\Delta}_a - \lambda)^{-1}$ is a meromorphic compact-operator-valued function of $\lambda$, and invoking the spectral theorem for self-adjoint compact operators. The compactness of the resolvent is a Rellich-type compactness result, proven by observing that $(\widetilde{\Delta}_a - \lambda)^{-1}$ maps $L^2_a$ to a Sobolev-type space $\mathfrak{B}^1_a$ with a finer topology on $\mathfrak{B}^1_a$ than the subspace topology and that the inclusion $\mathfrak{B}^1_a \to L^2_a$ is compact.

Chapter 11 uses the discretization results of Chapter 10 to prove meromorphic continuations and functional equations of a variety of Eisenstein series, following [Colin de Verdière 1981, 1982/1983]'s application of the discreteness result in [Lax-Phillips 1976]. This is carried out first for the four simple examples, then for maximal proper parabolic Eisenstein series for $SL_n(\mathbb{Z})$, with cuspidal data. In both the simplest cases and the higher-rank examples, we identify the *exotic eigenfunctions* as being certain truncated Eisenstein series.

Chapter 12 uses several of the analytical ideas and methods of the previous chapters to reconsider automorphic Green's functions, and solutions to other differential equations in automorphic forms, by spectral methods. We prove a *pretrace formula* in the simplest example, as an application of a comparably simple instance of a *subquotient theorem*, which follows from asymptotics of solutions of second-order ordinary differential equations, recalled in a later appendix. We recast the pretrace formula as a demonstration that an automorphic Dirac $\delta$-function lies in the expected *global automorphic Sobolev space*. The same argument gives a corresponding result for any compact automorphic *period*. Subquotient/subrepresentation theorems for groups such as $G = SO(n, 1)$ (rank-one groups with abelian unipotent radicals) appeared in [Casselman-Osborne 1975], [Casselman-Osborne 1978]. For higher-rank groups $SL_n(\mathbb{Z})$, the corresponding subrepresentation theorem is [Casselman 1978/1980], [Casselman-Miličić 1982]. Granting that, we obtain a corresponding pretrace formula for a class of compactly supported automorphic distributions, showing that these distributions lie in the expected global automorphic Sobolev spaces.

Chapter 13 is an extensive appendix with many examples of natural spaces of functions and appropriate topologies on them. One point is that too-limited types of topological vector spaces are inadequate to discuss natural function

spaces arising in practice. We include essential standard arguments characterizing locally convex topologies in terms of families of seminorms. We prove the *quasi-completeness* of all natural function spaces, weak duals, and spaces of maps between them. Notably, this includes spaces of distributions.

Chapter 14 proves existence of Gelfand-Pettis vector-valued integrals of compactly supported continuous functions taking values in locally convex, quasi-complete topological vector space. Conveniently, the previous chapter showed that all function spaces of practical interest meet these requirements. The fundamental property of Gelfand-Pettis integrals is that for $V$-valued $f$, $T : V \to W$ continuous linear,

$$T\Big(\int f\Big) = \int T \circ f$$

at least for $f$ continuous, compactly supported, $V$-valued, where $V$ is quasi-complete and locally convex. That is, continuous linear operators pass inside the integral. In suitably topologized natural function spaces, this situation includes differentiation with respect to a parameter. In this situation, as corollaries we can easily prove uniqueness of invariant distributions, density of smooth vectors, and similar.

Chapter 15 carefully discusses holomorphic $V$-valued functions, using the Gelfand-Pettis integrals as well as a variant of the Banach-Steinhaus theorem. That is, weak holomorphy implies (strong) holomorphy, and the expected Cauchy integral formulas and Cauchy-Goursat theory apply almost *verbatim* in the vector-valued situation. Similarly, we prove that for $f$ a $V$-valued function on an interval $[a, b]$, $\lambda \circ f$ being $C^k$ for all $\lambda \in V^*$ implies that $f$ itself is $C^{k-1}$ as a $V$-valued function.

Chapter 16 reviews basic results on asymptotic expansions of integrals and of solutions to second-order ordinary differential equations. The methods are deliberately general, rather than invoking specific features of special functions, to illustrate methods that are applicable more broadly. The simple subrepresentation theorem in Chapter 12 makes essential use of asymptotic expansions.

Our coverage of modern analysis does not aim to be either systematic or complete but well-grounded and adequate for the aforementioned issues concerning automorphic forms. In particular, several otherwise-apocryphal results are treated carefully. We want a sufficient viewpoint so that attractive heuristics, for example, from physics, can become succinct, genuine proofs. Similarly, we do *not* presume familiarity with Lie theory, nor algebraic groups, nor representation theory, nor algebraic geometry, and certainly not with classification of representations of Lie groups or $p$-adic groups. All these are indeed useful, in the long run, but it is unreasonable to demand mastery of these before

thinking about analytical issues concerning automorphic forms. Thus, we directly develop some essential ideas in these supporting topics, sufficient for immediate purposes here. [Lang 1975] and [Iwaniec 2002] are examples of the self-supporting exposition intended here.

Naturally, any novelty here is mostly in the presentation, rather than in the facts themselves, most of which have been known for several decades. Sources and origins can be most clearly described in a historical context, as follows.

The reduction theory in [1.5] is merely an imitation of the very classical treatment for $SL_2(\mathbb{Z})$, including some modern ideas, as in [Borel 1997]. The subtler versions in [2.2] and [3.3] are expanded versions of the first part of [Godement 1962–1964], a more adele-oriented reduction theory than [Borel 1965/1966b], [Borel 1969], and [Borel-HarishChandra 1962]. Proofs [1.9.1], [2.8.6], [3.10.1-2], [3.11.1] of convergence of Eisenstein series are due to Godement use similar ideas, reproduced for real Lie groups in [Borel 1965/1966a]. Convergence arguments on larger groups go back at least to [Braun 1939]'s treatment of convergence of Siegel Eisenstein series. Holomorphic Hilbert-Blumenthal modular forms were studied by [Blumenthal 1903/1904]. What would now be called degenerate Eisenstein series for $GL_n$ appeared in [Epstein 1903/1907]. [Picard 1882, 1883, 1884] was one of the earliest investigations beyond the elliptic modular case. Our notion of *truncation* is from [Arthur 1978] and [Arthur 1980].

Eigenfunction expansions and various notions of convergence are a pervasive theme here and have a long history. The idea that periodic functions should be expressible in terms of sines and cosines is at latest from [Fourier 1822], including what we now call the Dirichlet kernel, although [Dirichlet 1829] came later. Somewhat more generally, eigenfunction expansions for Sturm-Liouville problems appeared in [Sturm 1836] and [Sturm 1833a,b,1836a,b] but were not made rigorous until [Bôcher 1898/1899] and [Steklov 1898] (see [Lützen 1984]). Refinements of the spectral theory of ordinary differential equations continued in [Weyl 1910], [Kodaira 1949], and others, addressing issues of non-compactness and unboundedness echoing complications in the behavior of Fourier transform and Fourier inversion on the line [Bochner 1932], [Wiener 1933]. Spectral theory and eigenfunction expansions for integral equations, which we would now call compact operators [9.A], were recognized as more tractable than direct treatment of differential operators soon after 1900: [Schmidt 1907], [Myller-Lebedev 1907], [Riesz 1907], [Hilbert 1909], [Riesz 1910], [Hilbert 1912]. Expansions in spherical harmonics were used in the 18th century by S.P. Laplace and J.-L. Lagrange, and eventually subsumed in the representation theory of compact Lie groups [Weyl 1925/1926], and in eigenfunction expansions on Riemannian manifolds and Lie groups,

as in [Minakshisundaram-Pleijel 1949], [Povzner 1953], [Avakumović 1956], [Berezin 1956], and many others.

Spectral decomposition and synthesis of various types of automorphic forms is more recent, beginning with [Maaß 1949], [Selberg 1956], and [Roelcke 1956a, 1956b]. The spectral decomposition for automorphic forms on general reductive groups is more complicated than might have been anticipated by the earliest pioneers. Subtleties are already manifest in [Gelfand-Fomin 1952], and then in [Gelfand-Graev 1959], [Harish-Chandra 1959], [Gelfand-PiatetskiShapiro 1963], [Godement 1966b], [Harish-Chandra 1968], [Langlands 1966], [Langlands 1967/1976], [Arthur 1978], [Arthur 1980], [Jacquet 1982/1983], [Moeglin-Waldspurger 1989], [Moeglin-Waldspurger 1995], [Casselman 2005], [Shahidi 2010]. Despite various formalizations, spectral synthesis of automorphic forms seems most clearly understood in fairly limited scenarios: [Godement 1966a], [Faddeev 1967], [Venkov 1971], [Faddeev-Pavlov 1972], [Arthur 1978], [Venkov 1979], [Arthur 1980], [Cogdell-PiatetskiShapiro 1990], largely due to issues of convergence, often leaving discussions in an ambiguous realm of (nevertheless interesting) heuristics.

Regarding meromorphic continuation of Eisenstein series: our proof [2.B] for the case [2.9] of $GL_2$ is an adaptation of the Poisson summation argument from [Godement 1966a]. The essential idea already occurred in [Rankin 1939] and [Selberg 1940]. [Elstrodt-Grunewald-Mennicke 1985] treated examples including our example $SL_2(\mathbb{Z}[i])$, and in that context [Elstrodt-Grunewald-Mennicke 1987] treats special cases of the *period* computation of [2.C]. For Eisenstein series in rank one groups, compare also [Cohen-Sarnak 1980], which treats a somewhat larger family including our simplest four examples, and then [Müller 1996]. The minimal-parabolic example in [3.12] using Bochner's lemma [3.A] essentially comes from an appendix in [Langlands 1967/1976]. The arguments for the broader class of examples in Chapter 11 are adaptations of [Colin de Verdière 1981, 1982/1983], using discretization effects of pseudo-Laplacians from Chapter 10, which adapts the idea of [Lax-Phillips 1976]. Certainly one should compare the arguments in [Harish-Chandra 1968], [Langlands 1967/1976], [Wong 1990], and [Moeglin-Waldspurger 1995], the last of which gives a version of Colin de Verdière's idea due to H. Jacquet.

The discussion of group actions on function spaces in Chapter 6 is mostly very standard. Apparently the first occurrence of the Gelfand-Kazhdan criterion idea is in [Gelfand 1950]. An extension of that idea appeared in [Gelfand-Kazhdan 1975].

The arguments for discrete decomposition of cuspforms in Chapter 11 are adaptations of [Godement 1966b]. The discrete decomposition examples

for larger spaces of pseudo-cuspforms in Chapter 10 use the idea of [Lax-Phillips 1976]. The idea of this decomposition perhaps goes back to [Gelfand-Fomin 1952] and, as with many of these ideas, was elaborated on in the iconic sources [Gelfand-Graev 1959], [Harish-Chandra 1959], [Gelfand-PiatetskiShapiro 1963], [Godement 1966b], [Harish-Chandra 1968], [Langlands 1967/1976], and [Moeglin-Waldspurger 1989].

Difficulties with pointwise convergence of Fourier series of continuous functions, and problems in other otherwise-natural Banach spaces of functions, were well appreciated in the late 19th century. There was a precedent for constructs avoiding strictly pointwise conceptions of functions in the very early 20th century, when B. Levi, G. Fubini, and D. Hilbert used Hilbert space constructs to legitimize Dirichlet's minimization principle, in essence that a nonempty closed convex set should have a (unique) point nearest a given point not in that set. The too-general form of this principle is false, in that both existence and uniqueness easily fail in Banach spaces, in natural examples, but the principle is correct in Hilbert spaces. Thus, natural Banach spaces of pointwise-valued functions, such as continuous functions on a compact set with sup norm, do not support this minimization principle. Instead, Hilbert-space versions of continuity and differentiability are needed, as in [Levi 1906]. This idea was systematically developed by [Sobolev 1937, 1938, 1950]. We recall the $L^2$ Sobolev spaces for circles in [9.5] and for lines in [9.7] and develop various (global) automorphic versions of Sobolev spaces in Chapters 10, 11, and 12.

For applications to analytic number theory, automorphic forms are often constructed by *winding up* various simpler functions containing parameters, forming *Poincaré series* [Cogdell-PiatetskiShapiro 1990] and [Cogdell-PiatetskiShapiro-Sarnak 1991]. Spectral expansions are the standard device for demonstration of meromorphic continuation in the parameters, if it exists at all, which is a nontrivial issue [Estermann 1928], [Kurokawa 1985a,b]. For the example of automorphic Green's functions, namely, solutions to equations $(\Delta - s(s-1))u = \delta_w^{\mathrm{afc}}$ with invariant Laplacian $\Delta$ on $\mathfrak{H}$ and automorphic Dirac $\delta$ on the right, [Huber 1955] had considered such matters in the context of lattice-point problems in hyperbolic spaces, and, independently, [Selberg 1954] had addressed this issue in lectures in Göttingen. [Neunhöffer 1973] carefully considers the convergence and meromorphic continuation of a solution of that equation formed by winding up. See also [Elstrodt 1973]. The complications or failures of pointwise convergence of the spectral synthesis expressions can often be avoided entirely by considering convergence in suitable global automorphic Sobolev spaces described in Chapter 12. See [DeCelles 2012] and [DeCelles 2016] for developments in this spirit.

Because of the naturality of the issue and to exploit interesting idiosyncrasies, we pay considerable attention to invariant Laplace-Beltrami operators and their eigenfunctions. To have genuine proofs, rather than heuristics, Chapter 9 attends to rigorous notions of unbounded operators on Hilbert spaces [vonNeumann 1929], with motivation toward [vonNeumann 1931], [Stone 1929, 1932], [Friedrichs 1934/1935], [Krein 1945], [Krein 1947]. In fact, [Friedrichs 1934/1935]'s special construction [9.2] has several useful idiosyncracies, exploited in Chapters 10 and 11. Incidentally, the apparent fact that the typically naive treatment of many natural Laplace-Beltrami operators without boundary conditions does not lead to serious mistakes is a corollary of their *essential self-adjointness* [9.9], [9.10]. That is, in many situations, the naive form of the operator admits a *unique* self-adjoint extension, and this extension is the *graph closure* of the original. Thus, in such situations, a naive treatment is provably reasonable. However, the Lax-Phillips discretization device, and Colin de Verdière's use of it to prove meromorphic continuation of Eisenstein series and also to convert certain inhomogeneous differential equations to homogeneous ones, illustrate the point that *restrictions* of essentially self-adjoint operators need not remain essentially self-adjoint. With hindsight, this possibility is already apparent in the context of Sturm-Liouville problems [9.3].

The *global automorphic Sobolev spaces* of Chapter 12 already enter in important auxiliary roles as the spaces $\mathfrak{B}^1$, $\mathfrak{B}^1_a$ in Chapter 10's proofs of discrete decomposition of spaces of pseudo-cuspforms, and $\mathfrak{E}^1$ and $\mathfrak{E}^1_a$ in [11.7-11.11] proving meromorphic continuation of Eisenstein series. The basic estimate called a *pretrace formula* occurred as a precursor to *trace formulas*, as in [Selberg 1954], [Selberg 1956], [Hejhal 1976/1983], and [Iwaniec 2002]. The notion of global automorphic Sobolev spaces provides a reasonable context for discussion of automorphic Green's functions, other automorphic distributions, and solutions of partial differential equations in automorphic forms. The heuristics for Green's functions [Green 1828], [Green 1837] had repeatedly shown their utility in the 19th century. Differential equations $(-\Delta - \lambda)u = \delta$ related to Green's functions had been used by physicists [Dirac 1928a/1928b, 1930], [Thomas 1935], [Bethe-Peierls 1935], with excellent corroboration by physical experiments and are nowadays known as *solvable models*. At the time, and currently, in physics contexts they are rewritten as $((-\Delta + \delta) - \lambda)u = 0$, viewing $-\Delta + \delta$ as a perturbation of $-\Delta$ by a *singular potential* $\delta$, a mathematical idealization of a very short-range force. This was treated rigorously in [Berezin-Faddeev 1961]. The necessary systematic estimates on eigenvalues of integral operators use a *subquotient theorem*, which we prove for the four simple examples, as in that case the issue is about asymptotics of solutions of second-order differential equations, classically understood as recalled

in an appendix (Chapter 16). The general result is the *subrepresentation theorem* from [Casselman 1978/1980], [Casselman-Miličić 1982], improving the *subquotient theorem* of [HarishChandra 1954]. In [Varadarajan 1989] there are related computations for $SL_2(\mathbb{R})$.

In the discussion of natural function spaces in Chapter 13, in preparation for the vector-valued integrals of the following chapter, the notion of *quasi-completeness* proves to be the correct general version of completeness. The *incompleteness* of weak duals has been known at least since [Grothendieck 1950], which gives a systematic analysis of completeness of various types of duals. This larger issue is systematically discussed in [Schaefer-Wolff 1966/1999], pp. 147–148 and following. The significance of the compactness of the closure of the convex hull of a compact set appears, for example, in the discussion of vector-valued integrals in [Rudin 1991], although the latter does not make clear that this condition is fulfilled in more than Fréchet spaces and does not mention quasi-completeness. To apply these ideas to distributions, one might cast about for means to prove the compactness condition, eventually hitting on the hypothesis of quasi-completeness in conjunction with ideas from the proof of the Banach-Alaoglu theorem. Indeed, in [Bourbaki 1987] it is shown (by apparently different methods) that quasi-completeness implies this compactness condition. The fact that a bounded subset of a countable strict inductive limit of closed subspaces must actually be a bounded subset of one of the subspaces, easy to prove once conceived, is attributed to Dieudonne and Schwartz in [Horvath 1966]. See also [Bourbaki 1987], III.5 for this result. Pathological behavior of uncountable colimits was evidently first exposed in [Douady 1963].

In Chapter 14, rather than *constructing* vector-valued integrals as limits following [Bochner 1935], [Birkhoff 1935], et alia, we use the [Gelfand 1936]-[Pettis 1938] *characterization* of integrals, which has good functorial properties and gives a forceful reason for *uniqueness*. The issue is *existence*. Density of smooth vectors follows [Gårding 1947]. Another of application of holomorphic and meromorphic vector-valued functions is to *generalized functions*, as in [Gelfand-Shilov 1964], studying *holomorphically parametrized families* of distributions. A hint appears in the discussion of holomorphic vector-valued functions in [Rudin 1991]. A variety of developmental episodes and results in the Banach-space-valued case is surveyed in [Hildebrandt 1953]. Proofs and application of many of these results are given in [Hille-Phillips 1957]. (The first edition, authored by Hille alone, is sparser in this regard.) See also [Brooks 1969] to understand the viewpoint of those times.

Ideas about vector-valued holomorphic and differentiable functions, in Chapter 15, appeared in [Schwartz 1950/1951], [Schwartz 1952], [Schwartz 1953/1954], and in [Grothendieck 1953a,1953b].

The asymptotic expansion results of Chapter 16 are standard. [Blaustein-Handelsman 1975] is a standard source for asymptotics of integrals. *Watson's lemma* and *Laplace's method* for integrals have been used and rediscovered repeatedly. Watson's lemma dates from at latest [Watson 1918], and Laplace's method at latest from [Laplace 1774]. [Olver 1954] notes that Carlini [Green 1837] and [Liouville 1837] investigated relatively simple cases of asymptotics at irregular singular points of ordinary differential equations, without complete rigor. According to [Erdélyi 1956] p. 64, there are roughly two proofs that the standard argument produces genuine asymptotic expansions for solutions of the differential equation. Poincaré's approach, elaborated by J. Horn, expresses solutions as Laplace transforms and invokes Watson's lemma to obtain asymptotics. G.D. Birkhoff and his students constructed auxiliary differential equations from partial sums of the asymptotic expansion, and compared these auxiliary equations to the original [Birkhoff 1908], [Birkhoff 1909], [Birkhoff 1913]. Volterra integral operators are important in both approaches, insofar as asymptotic expansions behave better under *integration* than under *differentiation*. Our version of the Birkhoff argument is largely adapted from [Erdélyi 1956].

Many parts of this exposition are adapted and expanded from [Garrett vignettes], [Garrett mfms-notes], [Garrett fun-notes], and [Garrett alg-noth-notes]. As is surely usual in book writing, many of the issues here had plagued me for decades.