1

Four Small Examples

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We recall basic notions related to automorphic forms on some simple arithmetic quotients, including the archetypical quotient \( SL_2(\mathbb{Z})\backslash \mathfrak{H} \) of the complex upper half-plane \( \mathfrak{H} \) and the related \( SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}) \). To put this in a somewhat larger context, we consider parallel examples \( \Gamma \backslash X \) and \( \Gamma \backslash G \) for a few other groups \( G \), discrete subgroups \( \Gamma \), and spaces \( X \approx G/K \) for compact subgroups \( K \) of \( G \). The

\[ \text{In slightly more sophisticated terms inessential to this discussion: the four examples } G \text{ immediately considered are real-rank one semi-simple Lie groups, and the discrete subgroups } \Gamma \text{ are unicuspidal in the sense that } \Gamma \backslash G/K \text{ is reasonably compactified by adding just a single cusp, where } K \text{ is a (maximal) compact subgroup of } G. \text{ That is, the reduction theory of } \Gamma \backslash G \text{ is especially simple in these four cases. Examples with larger real rank, such as } GL_n \text{ with } n \geq 3, \text{ are considered later.} \]
other three examples share several of the features of $G = \text{SL}_2(\mathbb{R})$, $\Gamma = \text{SL}_2(\mathbb{Z})$, and $X = \mathcal{H} \cong G/K$ with $K = \text{SO}_2(\mathbb{R})$, allowing simultaneous treatment.

For many reasons, even if we are only interested in harmonic analysis on quotients $\Gamma \backslash X$, it is necessary to consider spaces of functions on the overlying spaces $\Gamma \backslash G$, on which $G$ acts by right translations, with a corresponding translation action on functions.

Some basic discussions not specific to the four examples are postponed, such as determination of invariant Laplacians in coordinates, self-adjointness properties of invariant Laplacians, proof of the formula for the left $G$-invariant measure on $X = G/K$, unwinding properties of integrals and sums, continuity of the action of $G$ on test functions on $\Gamma \backslash G$, density of test functions in $L^2(\Gamma \backslash X)$, vector-valued integrals, holomorphic vector-valued functions, and other generalities.

We also postpone the relatively specific proofs of the major theorems stated in the final sections of this chapter, concerning the spectral decomposition of automorphic forms, meromorphic continuation of Eisenstein series, and the theory of the constant term. Those proofs make pointed use of finer details from the more sophisticated analysis.

### 1.1 Groups $G = \text{SL}_2(\mathbb{R}), \text{SL}_2(\mathbb{C}), \text{Sp}^*_1, \text{SL}_2(\mathbb{H})$

These four groups share some convenient simplifying features, which we will exploit. The first two examples $G$ are easy to describe:

$$G = \begin{cases} 
\text{SL}_2(\mathbb{R}) = \text{a special linear group over } \mathbb{R} \\
= \text{two-by-two determinant-1 real matrices} \\
\text{SL}_2(\mathbb{C}) = \text{a special linear group over } \mathbb{C} \\
= \text{two-by-two determinant-1 complex matrices}
\end{cases}$$

We will have occasion to use the general linear groups $\text{GL}_2(R)$ of 2-by-2 invertible matrices with entries in a ring $R$. Our other two example groups are conveniently described in terms of the Hamiltonian quaternions $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, with the usual relations

$$i^2 = j^2 = k^2 = -1 \quad ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j$$

The quaternion conjugation is $\overline{\alpha} = a + bi + cj + dk = a - bi - cj - dk$ for $\alpha = a + bi + cj + dk$, the norm is $N\alpha = \alpha \cdot \overline{\alpha}$, and $|\alpha| = (N\alpha)^{\frac{1}{2}}$. $\mathbb{H}$ can be modeled in two-by-two complex matrices by

$$\rho(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\
-c + di & a - bi \end{pmatrix}$$
1.1 Groups $G = \text{SL}_2(\mathbb{R}), \text{SL}_2(\mathbb{C}), \text{Sp}^\ast_{1,1}$, and $\text{SL}_2(\mathbb{H})$

with $\det \rho(\alpha) = N\alpha$. For a quaternion matrix $g$, let $g^*$ be the transpose of the entry-wise conjugate:

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}^* = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\
\overline{\beta} & \overline{\delta}
\end{pmatrix} \quad \text{(for } \alpha, \beta, \gamma, \delta \in \mathbb{H})
$$

The third example group is a kind of symplectic group: letting $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, this group is

$$
G = \text{Sp}^\ast_{1,1} = \{ g \in \text{GL}_2(\mathbb{H}) : g^* S g = S \}
$$

The fourth example is a special linear group $G = \text{SL}_2(\mathbb{H})$. In the latter, $\text{SL}_2$ is more convenient than $\text{GL}_2$, having a smaller center. However, because $\mathbb{H}$ is not commutative, the notion of determinant is problematic. One way to skirt the issue is to imbed $r : \text{GL}_2(\mathbb{H}) \to \text{GL}_4(\mathbb{C})$; with quaternions $\alpha, \beta, \gamma, \delta$,

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \begin{pmatrix} \rho(\alpha) & \rho(\beta) \\
\rho(\gamma) & \rho(\delta) \end{pmatrix}
$$

identified with a 4-by-4 complex matrix, using the map $\rho$ of $\mathbb{H}$ to 2-by-2 complex matrices, and require that the image in $\text{GL}_4(\mathbb{C})$ be in the subgroup $\text{SL}_4(\mathbb{C})$ where determinant is 1:

$$
\text{SL}_2(\mathbb{H}) = \{ g \in \text{GL}_2(\mathbb{H}) : r(g) \in \text{SL}_4(\mathbb{C}) \}
$$

Standard subgroups of any of these groups $G$ are

$$
P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad M = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}
$$

$$
A^+ = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \}
$$

The Levi-Malcev decomposition $P = NM$ is elementary to check. By direct computation from the defining relations of the groups, one finds

$$
M = \begin{cases}
\left\{ \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} : m \in \mathbb{R}^\times \right\} & \text{(for } G = \text{SL}_2(\mathbb{R}) \text{)} \\
\left\{ \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} : m \in \mathbb{C}^\times \right\} & \text{(for } G = \text{SL}_2(\mathbb{C}) \text{)} \\
\left\{ \begin{pmatrix} m & 0 \\ 0 & \overline{m}^{-1} \end{pmatrix} : m \in \mathbb{H}^\times \right\} & \text{(for } G = \text{Sp}^\ast_{1,1} \text{)} \\
\left\{ a \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} : N(ad) = 1, \ a, d \in \mathbb{H}^\times \right\} & \text{(for } G = \text{SL}_2(\mathbb{H}) \text{)}
\end{cases}
$$
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and

\[ N = \begin{cases} 
\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} & \text{(for } G = SL_2(\mathbb{R})\text{)} \\
\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{C} \right\} & \text{(for } G = SL_2(\mathbb{C})\text{)} \\
\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{H}, x + \bar{x} = 0 \right\} & \text{(for } G = Sp^*_1, 1\text{)} \\
\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{H} \right\} & \text{(for } G = SL_2(\mathbb{H})\text{)} 
\end{cases} \]

The subgroup \( P \) is the standard (proper) parabolic, \( N \) is its unipotent radical, \( M \) is the standard Levi-Malcev component, and \( A^+ \) is the standard split component. We will use these (standard) names without elaborating on their history or their connotations.

In these examples, the (spherical) Bruhat decomposition is

\[ G = \bigcup_{w=1, w_0} P w_0 P = P \downarrow P w_0 P = P \downarrow P w_0 N \]

where \( w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), with the last equality following because \( w_0 \) normalizes \( M \):

\[ P w_0 P = P w_0 M N = P (w_0 M w_0^{-1}) w_0 N = P w_0 N \]

The element \( w_0 \) is the long Weyl element. The small (Bruhat) cell is \( P \) itself, and the big (Bruhat) cell is \( P w_0 P \). The (spherical, geometric) Weyl group is \( \{ 1, w_0 \} \).

It is a group modulo the center of \( G \). The proof of the Bruhat decomposition is straightforward: \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P \) if and only if \( c = 0 \). Otherwise, \( c \neq 0 \), and we try to find \( p \in P \) and \( n \in N \) such that \( g = p w_0 n \). To simplify, since \( c \neq 0 \), it is invertible, so, in a form applicable to all four cases, we can left multiply by \( \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \in M \) to make \( c = 1 \) without loss of generality. Then try to solve

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g = p w_0 n = \begin{pmatrix} p_{11} & p_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & n_{12} \\ 0 & 1 \end{pmatrix} 
\]

\[
= \begin{pmatrix} p_{12} & p_{12} n_{12} - p_{11} \\ 0 & n_{12} \end{pmatrix}
\]
1.2 Compact Subgroups $K \subset G$, Cartan Decompositions

From the lower right entry, apparently $a_{12} = d$. For the case $G = Sp^*_1$, the additional condition must be checked, as follows. Observe that inverting $g^*Sg = S$ gives $g^{-1}S^{-1}(g^*)^{-1} = S^{-1}$, and then $S = gSg^*$. In particular, this gives a relation between the $c, d$ entries of $g$:

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = S = gSg^* = \begin{pmatrix}
* & * \\
* & *
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
* & c \\
* & d\end{pmatrix} = \begin{pmatrix}
* & * \\
* & cd + dc
\end{pmatrix}
$$

For $c = 1$, this gives $d + d = 0$, which is the condition for $\begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix} \in N$ in that case. Thus, in all cases, right multiplying $g$ by $\begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix} \in N$ makes $d = 0$, without loss of generality. Thus, it suffices to solve

$$
\begin{pmatrix}
a & b \\
1 & 0
\end{pmatrix} = g = pw_o = \begin{pmatrix}
p_{11} & p_{12} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
p_{12} & -p_{11} \\
1 & 0
\end{pmatrix}
$$

That is,

$$
gw_o^{-1} = \begin{pmatrix}
-b & a \\
0 & 1
\end{pmatrix} = p
$$

Since $g \in G$, the entries $a, b$ satisfy whatever relations $G$ requires, and $p \in G$. This proves the Bruhat decomposition.

1.2 Compact Subgroups $K \subset G$, Cartan Decompositions

We describe the standard maximal compact subgroups $K \subset G$ for the four examples $G$. With $\mathbb{H}^1$ the quaternions of norm 1, in a notation consistent with that for $Sp^*_1$, write

$$
Sp^*_1 = \{g \in GL_1(\mathbb{H}) : g^*g = 1\} = \{g \in \mathbb{H}^\times : \overline{g}g = 1\} = \mathbb{H}^1
$$

Letting $I_2$ be the two-by-two identity matrix, the four maximal compact subgroups are

$$
K = \left\{ \begin{array}{ll}
SO_2(\mathbb{R}) &= \{g \in SL_2(\mathbb{R}) : g^\top g = 1_2\} \quad \text{(for } G = SL_2(\mathbb{R})\} \\
SU_2 &= \{g \in SL_2(\mathbb{C}) : g^*g = 1_2\} \quad \text{(for } G = SL_2(\mathbb{C})\} \\
Sp^*_1 \times Sp^*_1 &= \mathbb{H}^1 \times \mathbb{H}^1 \quad \text{(for } G = Sp^*_1,1\} \\
Sp^*_2 &= \{g \in GL_2(\mathbb{H}) : g^*g = 1_2\} \quad \text{(for } G = SL_2(\mathbb{H})\}
\end{array} \right.
$$

\(^2\) The maximality of each of these subgroups $K$ among all compact subgroups in the corresponding $G$ is not obvious but is not used in the sequel.
In all four cases, the indicated groups are compact. Verification of the compactness of the first three is straightforward because their defining equations present them as spheres or products of spheres. Verification that $S\rho_2^4$ is compact and is a subgroup of $SL_2(\mathbb{H})$ merits discussion. For the fourth, observe that the defining condition

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^*\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
|a|^2 + |c|^2 & \overline{ab} + \overline{cd} \\
\overline{ba} + \overline{dc} & |b|^2 + |d|^2
\end{pmatrix}
$$

makes $S\rho_2^4$ a closed subset of a product of two seven-spheres, $|a|^2 + |c|^2 = 1$ and $|b|^2 + |d|^2 = 1$, thus, compact. Further, $S\rho_2^4$ lies inside $SL_2(\mathbb{H})$ rather than merely $GL_2(\mathbb{H})$. For the moment, we will prove a slightly weaker property, that the relevant determinant is $\pm 1$. Use the feature

$$
\rho(\overline{\alpha}) = \epsilon \rho(\alpha)^\top \epsilon^{-1}
$$

(where $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, for $\alpha \in \mathbb{H}$) of the imbedding $\rho$ of $\mathbb{H}$ in 2-by-2 complex matrices, and again let

$$
\rho\begin{pmatrix}a & b \\
c & d\end{pmatrix} = \begin{pmatrix} \rho(a) & \rho(b) \\
\rho(c) & \rho(d) \end{pmatrix}
$$

(for $a, b, c, d \in \mathbb{H}$) viewed as mapping to 4-by-4 complex matrices. Then $r(g^*) = J \cdot r(g) \cdot J^{-1}$, where

$$
J = \begin{pmatrix} \epsilon & 0 \\
0 & \epsilon \end{pmatrix}
\begin{pmatrix} 1 & -1 \\
-1 & 1 \end{pmatrix}
$$

and $g \in GL_2(\mathbb{H})$. Thus, for $g^*g = 1_2 \in GL_2(\mathbb{H})$,

$$
1_4 = r(1_2) = r(g^*g) = r(g^*) \cdot r(g) = J \cdot r(g)^\top \cdot J^{-1} \cdot r(g)
$$

In other words, $r(g)^\top J r(g) = J$. Taking determinants shows $\det r(g)^2 = 1$, so $\det r(g) = \pm 1$. Thus, $g$ in the connected component of $S\rho_2^4$ containing $1$ has $\det r(g) = 1$.

The copy $K$ of $S\rho_1^4 \times S\rho_1^4$ inside $S\rho_{1,1}^4$ is not immediately visible in these coordinates, which were chosen to make the parabolic $P$ visible. That is, defining $S\rho_{1,1}^4$ as the isometry group of the quaternion Hermitian form $S$ obscures the nature of the (maximal) compact $K$. Changing coordinates by replacing $S$ by

$$
S' = \frac{1}{2} \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix} S \begin{pmatrix} 1 & -1 \\
1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\
1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\
0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\
-1 & 1 \end{pmatrix}
$$

3 Thus, $r(g)$ is inside a symplectic group denoted $Sp_4(\mathbb{C})$ or $Sp_2(\mathbb{C})$, depending on convention.
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gives
\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
SP_{1,1}^* \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}^{-1} = \{ g \in GL_2(\mathbb{H}) : g^*Sg = S' \}
\]
and makes the two copies of $SP_{1,1}^*$ visible on the diagonal:
\[
\begin{aligned}
\{ k = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} : k^*S'k = S' \} &= \{ k = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \}
\end{aligned}
\]
That is,
\[
K = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \right\} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]
\[
= \left\{ \begin{pmatrix} \mu + \nu & -\mu + \nu \\ -\mu + \nu & \mu + \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 \right\}
\]

[1.2.1] Claim: $K \cap P = K \cap M$, and

\[
K \cap M = \begin{cases}
\pm 1_2 & \text{(for } G = SL_2(\mathbb{R})) \\
\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : \mu \in \mathbb{C}^\times, \ |\mu| = 1 & \text{(for } G = SL_2(\mathbb{C})) \\
\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} : \mu \in \mathbb{H}^1 & \text{(for } G = SP_{1,1}^*) \\
\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} : \mu, \nu \in \mathbb{H}^1 & \text{(for } G = SL_2(\mathbb{H}))
\end{cases}
\]

Proof: In all but the third case, this follows from the description of $K$. For example, for $G = SL_2(\mathbb{R})$ and $K = SO_2(\mathbb{R})$, take $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$ and examine the relation $p^*p = 1_2$ for $p$ to be in $K$:

\[
p^*p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a^2 & (a + a^{-1})b \\ (a + a^{-1})b & b^2 + a^{-2} \end{pmatrix}
\]

From the upper-left entry, $a = \pm 1$. From the off-diagonal entries, $b = 0$. The arguments for $SL_2(\mathbb{C})$ and $SL_2(\mathbb{H})$ are similar. For $SP_{1,1}^*$, comparison to the
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coordinates that diagonalize $K \approx Sp^*_1 \times Sp^*_1$ gives

$$\left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{H}^1 \right\} = \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) K \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)^{-1}$$

$$\ni \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ 0 & (a^*)^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)^{-1}$$

$$= \frac{1}{2} \left( a + (a^*)^{-1} - b - a - (a^*)^{-1} + b \right) \left( a - (a^*)^{-1} - b + a + (a^*)^{-1} + b \right)$$

For example, adding the elements of the bottom row gives $a = k_2 \in \mathbb{H}^1$ and also $(a^*)^{-1} = a$. From either off-diagonal entry, $b = 0$.

In all four cases, the same discussion gives $M = A^+ \cdot (P \cap K) = A^+ \cdot (M \cap K)$.

The following will be essential in [7.1]:

[1.2.2] Claim: (Cartan decomposition) $G = KA^+ K$.

Proof: First, treat $G = SL_2(\mathbb{R})$. Prove that every $g \in G$ can be written as $g = sk$ with $s^\top = s$ and $s$ positive-definite. To find such $s$, assume for the moment that it exists, and consider

$$g \cdot g^\top = (sk) \cdot (sk)^\top = sk \cdot k^{-1} s = s^2$$

Certainly $gg^\top$ is symmetric and positive-definite, so having a positive-definite symmetric square root of positive-definite symmetric $t$ would produce $s$. Such $t$ gives a positive, symmetric operator on $\mathbb{R}^2$, which by the spectral theorem has an orthonormal basis of eigenvalues. That is, there is $h \in K$ such that $hh^\top = \delta$ is diagonal, necessarily with positive diagonal entries. With $\delta^{\frac{1}{2}}$ being the positive diagonal square root of $\delta$,

$$(h^\top \delta^{\frac{1}{2}} h)^2 = h^\top \delta^{\frac{1}{2}} h \cdot h^\top \delta^{\frac{1}{2}} h = h^\top \delta^{\frac{1}{2}} \cdot \delta^{\frac{1}{2}} h = h^\top \delta h = t$$

Thus, take $s = h^\top \delta^{\frac{1}{2}} h$, and every $g \in G$ can be written as $g = sk$. Indeed, we have more:

$$g = ks = k \cdot h^\top \delta^{\frac{1}{2}} h = (k \cdot h^\top) \cdot \delta^{\frac{1}{2}} \cdot h \in K \cdot A^+ \cdot K$$

giving the claim in this case. The cases of $G = SL_2(\mathbb{C})$ is similar, using $g = sk$ with $s = s^*$ Hermitian positive-definite and $k^* = k^{-1} \in K$, invoking the spectral theorem for Hermitian positive-definite operators. The same argument succeeds for $G = SL_2(\mathbb{H})$ with quaternion conjugation replacing complex, with a
1.3 Iwasawa Decomposition $G = PK = NA^+ K$

suitably adapted spectral theorem for $s \in GL_2(\mathbb{H})$ with $s^* = s$ and $x^* sc$ real and positive for all nonzero 2-by-1 quaternion matrices $x$.\footnote{4} The case of $G = Sp_{1,1}^*$ essentially reduces to the case of $SL_2(\mathbb{H})$, as follows. Since $g^* Sg = S$, $SgS^{-1} = (g^*)^{-1}$. Anticipating the Cartan decomposition $g = sk$, from $gg^* = ss^* = s^2$, by the quaternionic version of the spectral theorem, there is $k \in Sp_{1,1}^*$ such that $k^{-1} gg^* k = \Lambda$ with $\Lambda$ positive real diagonal. We want to adjust $k$ to be in $Sp_{1,1}^* \cap Sp_{2,2}^*$, while preserving the property $k^{-1} gg^* k = \Lambda$. Unless $gg^*$ is scalar, the diagonal entries are distinct. By $SgS^{-1} = (g^*)^{-1}$ and $Sg^* S^{-1} = g^{-1}$ for $g \in G$,

$$
\Lambda^{-1} = (\Lambda^*)^{-1} = S \Lambda S^{-1} = S(k^{-1} gg^* k) S^{-1} = (SkS^{-1})^{-1} \cdot Sg^* S^{-1} \cdot SkS^{-1} = (SkS^{-1})^{-1} \cdot (gg^*)^{-1} \cdot SkS^{-1}
$$

Inverting gives $\Lambda = (SkS^{-1})^{-1} \cdot gg^* \cdot SkS^{-1}$. Also $\Lambda = k^{-1} gg^* k$, so

$$(SkS^{-1}) \cdot \Lambda \cdot (SkS^{-1})^{-1} = gg^* = k \cdot \Lambda \cdot k^{-1}$$

That is, $k^{-1} \cdot SkS^{-1}$ commutes with $\Lambda$, and $\delta = k^{-1} \cdot SkS^{-1}$ is at worst diagonal:

$$
SkS^{-1} = k \cdot \delta = k \cdot \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}
$$

Since $\delta \in Sp_{2,2}^*$, $a \cdot \bar{a} = 1$ and $d \cdot \bar{d} = 1$. To preserve $k^{-1} gg^* k = \Lambda$, to adjust $k$ to be in $K = Sp_{1,1}^* \cap Sp_{2,2}^*$, adjust $k$ by diagonal matrices $\epsilon \in Sp_{2,2}^*$. The condition for $k \epsilon$ to be in $K$ is

$$(k \cdot \epsilon) = ((ke)^*)^{-1} = S(ke)S^{-1} = SkS^{-1} \cdot \epsilon S^{-1} = k \cdot \delta \cdot \epsilon S^{-1}$$

so take $\epsilon = S^{-1} \delta S$. The rest of the argument runs as in the first three cases. ///

1.3 Iwasawa Decomposition $G = PK = NA^+ K$

The subgroups $P$ and $K$ are not normal in $G$, so the Iwasawa decompositions $G = PK = \{pk : p \in P, k \in K\}$ do not express $G$ as a product group. Nevertheless, these decompositions are essential.

[1.3.1] Claim: (Iwasawa decomposition) $G = PK = NA^+ K$. In particular, the map $N \times A^+ \times K \rightarrow G$ by $n \times a \times k \rightarrow n ak$ is an injective set map (and is a diffeomorphism).

\footnote{4} In all three of these cases, a Rayleigh-Ritz approach gives a sufficient spectral theorem, as follows. Let $F$ be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $(x, y) = y^* x$ for 2-by-1 matrices $x, y$ over $F$. Let $T : F^2 \rightarrow F^2$ be right $F$-linear and positive Hermitian in the sense that $(Tx, x)$ is positive, real for $x \neq 0$. Then $x$ with $(x, x) = 1$ maximizing $(Tx, x)$ is an eigenvector for $T$. For nonscalar $T$, the unit vector $y$ minimizing $(Ty, y)$ is an eigenvector for $T$ orthogonal to $x$. Letting $k$ be the matrix with columns $x, y$, the conjugated matrix $k^{-1} Tk$ is diagonal.
**Four Small Examples**

**Proof:** For \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G \), in the easy case that \( c = 0 \), then \( g \in P \). In all cases, once we have \( g = nm \in P \), we can adjust \( g \) on the right by \( M \cap K \) to put the Levi component \( m \) into \( A^+ \).

One approach is to think of right multiplication by \( K \) as rotating the lower row \((c,d)\) of \( g \in G \) to put it into the form \((0,\ast)\) of the lower row of an element of \( P \). For \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G = SL_2(\mathbb{R}) \): right multiplication by the explicit element

\[
 k = \left( \begin{array}{cc} d & c \\ \sqrt{c^2 + d^2} & -c \\ \sqrt{c^2 + d^2} & d \end{array} \right) \in K = SO_2(\mathbb{R})
\]

puts \( gk \in P \):

\[
 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} d & c \\ \sqrt{c^2 + d^2} & -c \\ \sqrt{c^2 + d^2} & d \end{array} \right) = \left( \begin{array}{cc} \ast & \ast \\ 0 & \ast \end{array} \right)
\]

Similarly, for \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G = SL_2(\mathbb{C}) \), right multiplication by

\[
 k = \left( \begin{array}{cc} d & c \\ \sqrt{|c|^2 + |d|^2} & -c \\ \sqrt{|c|^2 + |d|^2} & d \end{array} \right) \in K = SU(2)
\]

gives \( gk \in P \). Likewise, for \( G = SL_2(\mathbb{H}) \), nearly the same explicit expression as for \( SL_2(\mathbb{C}) \) succeeds, with complex conjugation replaced by quaternion conjugation, accommodating the noncommutativity.

\[
 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} c^{-1}d & 1 \\ \sqrt{1 + |c^{-1}d|^2} & \sqrt{1 + |c^{-1}d|^2} \\ -1 & c^{-1}d \end{array} \right) = \left( \begin{array}{cc} \ast & \ast \\ 0 & \ast \end{array} \right) \in P
\]

For \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G = Sp^*_1 \), we hope that a matrix \( k \) of a similar form lies in \( K \approx Sp^*_1 \times Sp^*_1 \), and then \( gk \in P \). To be sure that the defining relation for

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5 This explicit element lies in the connected component of \( Sp^*_2 \) containing 1, so this argument for the Iwasawa decomposition is complete whether or not we have verified that \( Sp^*_2 \subset SL_2(\mathbb{H}) \).