# 1

# Probabilizing Fibonacci Numbers

Persi Diaconis

## Abstract

The question, "What does a typical Fibonacci number look like?" leads to interesting (and impossible) mathematics and many a tale about Ron Graham.

I have talked to Ron Graham every day, more or less, for the past 43 years. Some days we miss a call but some days it's three or four calls; so "once a day" seems about right. There are many reasons: he tells bad jokes, solves my math problems, teaches me things, and is my pal. The following tries to capture a few minutes with Ron.

## 1.1 A Fibonacci Morning

This term I'm teaching undergraduate number theory at Stanford's Department of Mathematics. It's a course for beginning math majors and the start is pretty dry (I'm in Week 1): every integer is divisible by a prime; if $p$ divides $ab$ then either $p$ divides $a$ or $p$ divides $b$; unique factorization. I'm using Bill LeVeque's fine *Fundamentals of Number Theory*. It's clear, correct, and cheap (a Dover paperback). He mentions the Fibonacci numbers and I decide to spend some time there to liven things up.

Fibonacci numbers have a "crank math" aspect but they are also serious stuff – from sunflower seeds through Hilbert's tenth problem. My way of understanding anything is to ask, "'What does a typical one look like?" Okay, what does a typical Fibonacci number look like? How many are even? What about the decomposition into prime powers? Are there infinitely many prime Fibonacci numbers? I realize that I don't know and, it turns out, for most of these questions, nobody knows. This is two hours before class and I do what I always do: call Ron.

"Hey, do we know what proportion of Fibonacci numbers are even?"

"Sure," he says without missing a beat. "It's 1/3, and it's easy: let me explain. If you write them out,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots,$$

you see that every third one is even, and it's easy to see from the recurrence." Similarly, every fourth one is a multiple of three, so 1/4 are divisible by 3. I note that the period for five is 5, ruining my guess at the pattern. He tells me that after five, the period for the prime $p$ is a divisor of $p \pm 1$; so 1/8 of all the $F_n$ are divisible by seven. Actually, things are a bit more subtle. For any prime $p$, the *sequence* of Fibonacci numbers (mod $p$) is periodic. Let's start them at $F_0 = 0$, when $p = 3$:

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, \ldots.$$

The length of the period is called the *Pisano period* (with its own Wikipedia page). The period of three is 8 and there are two zeros, so 1/4 are divisible by 3; 3/8 are 1 (mod 3) and 3/8 are 2 (mod 3). These periods turn out to be pretty chaotic and much is conjectural. The rest of my questions, e.g., is $F_n$ prime infinitely often, are worse: "not in our lifetimes," (Ron says) Erdős said.

Going back to my phone conversation with Ron, he says:

> Here's something your kids can do: You know the Fibonacci numbers grow pretty fast. This means that $\sum_{n=0}^{\infty} 1/F_{2^n}$ converges to its limiting value very fast. It turns out to be a quadratic irrational (!) *and* you can show that if a number has a more rapidly converging rational approximation, it's transcendental. (!)

And then he says, "Here's an easier one for your kids: ask them to add up $F_n/10^n$, $1 \leq n < \infty$." Answer: 10/89 (!)

It turns out that Ron had worked on my questions before. A 1964 paper [7] starts, "Let $S(L_0, L_1) = L_0, L_1, L_2, \ldots$ be the sequence of integers which satisfy the recurrence $L_{n+2} = L_{n+1} + L_n$, $n = 0, 1, 2, \ldots$. It is clear that the values $L_0$ and $L_1$ determine $S(L_0, L_1)$, e.g., $L(0, 1)$ is just the sequence of Fibonacci numbers. It is not known whether or not infinitely many primes occur in $S(0, 1)$ ...." He goes on to find an opposite: a pair $L_0, L_1$ so that *no* primes occur in $S(L_0, L_1)$. His best solution was

$$L_0 = 331635635998274737472200656430763$$
$$L_1 = 1510028911088401971189590305498785$$

This was subsequently improved by Knuth and then Wilf. The problem itself now has its own Wikipedia page; search for "primefree sequence." The current record is

$$L_0 = 106276436867$$
$$L_1 = 35256392432$$

due to Vesemirsov.

Finding such sequences is related to problems such as, “Is every odd number the sum of a prime plus a power of 2?” The answer is no; indeed Erdös [6] found arithmetic progressions with no numbers of this form. For this, he created the topic/tool of “covering congruences”: a sequence $\{a_1 \pmod{n_1}, \ldots, a_k \pmod{n_k}\}$ of finitely many residue classes $\{a_i + n_i x, x \in \mathbb{Z}\}$ whose union covers $\mathbb{Z}$. For example, $\{0 \pmod 2, 0 \pmod 3, 1 \pmod 4, 5 \pmod 6, 7 \pmod{12}\}$ is a covering set where all moduli are distinct. Erdős asked if there were such distinct covering systems where the smallest modulus – two in the example – is arbitrarily large. A variety of number theory hackers found examples. For instance, Nicesend found a set of more than $10^{50}$ distinct congruences with minimum modulus 40. One of Ron’s favorite negative results is a theorem of Hough [9]: there is an absolute upper bound to the minimum modulus of a system of distinct covering congruences. The Wikipedia phrase is “covering sequences.”

The preceding paragraphs are amplified from sentences of this same phone conversation. Ron has worked on math problems where Fibonacci facts form a crucial part of the argument “from then to now.” For example, in joint work with Fan Chung [2] they solved an old conjecture of D. J. Newman: for a sequence of numbers (mod 1), $x = (x_0, x_1, x_2, \ldots)$, define the strong discrepancy

$$D(x) = \inf_n \inf_m n\|x_n - x_{n+m}\|.$$

They found the following:

**Theorem 1.1.**

$$\sup_x D(x) \leq \frac{1}{\sqrt{5}}.$$

The reader who looks will find Fibonacci numbers throughout the proofs; $\sum_{n=0}^{\infty} 1/F_{2n}$ makes an appearance.

As a parting shot in our conversation, Ron moved back to the periods of Fibonacci and Lucus sequences $SL(L_0, L_1)$ in the preceding text. “You know, we had a pretty good trick in our book [4] using Fibonacci periods. You should perform it for your kids.” Let me perform it for you. To understand the connection, see Diaconis and Graham [4, p. 187].

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Excerpt
More Information



The performer draws a $4 \times 4$ square on a sheet of paper. A prediction is written on the back (to own up, it's 49).

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The patter goes as follows: "They teach kids the craziest things in school nowadays. The other day my daughter came home talking about 'adding mod seven.' That means you add and take away anything over 7, so $5 + 5 = 10 = 3 \pmod 7$. Here, let's try it out." Pick any two small numbers; say 5 and 6 are chosen. Write them down in positions $(1, 1)$ and $(1, 2)$. Then sum mod 7 in position $(1, 3)$:

| | | | |
|---|---|---|---|
| 5 | 6 | 4 | |
| | | | |
| | | | |
| | | | |

$\longrightarrow$

| | | | |
|---|---|---|---|
| 5 | 6 | 4 | 3 |
| 0 | 3 | 3 | 6 |
| | | | |
| | | | |

$\longrightarrow$

| | | | |
|---|---|---|---|
| 5 | 6 | 4 | 3 |
| 0 | 3 | 3 | 6 |
| 2 | 1 | 3 | 4 |
| 0 | 4 | 4 | 1 |

Continue as shown, adding successive pairs row by row until all squares are filled. You can ask spectators to help along the way. At the end, have someone (carefully) add up all 16 numbers in the usual way. The sum will match your prediction, 49. Our write-up gives pointers to the mathematical literature on Fibonacci periods.

In a follow-up call, I mentioned a charming fact pointed out by Susan Holmes. If you want to convert from miles to kilometers (and back) take the next Fibonacci number (or the one before, to go back). Thus 5 miles is close to 8 kilometers, 13 miles is close to 21 kilometers, 144 kilometers is close to 89 miles, and so on. To do general numbers, use Zeckendorf's theorem: any positive integer can be represented as a sum of distinct Fibonacci numbers: uniquely, if you never use two consecutive $F_n$. So $100 = 89 + 8 + 3$, and 100 miles is about $144 + 13 + 5 = 162$ kilometers. (Really, 100 miles $= 160.934$ kilometers; it's only an approximation.) The Zeckendorf representation is easy to find, just subtract off the largest possible $F_n$ each time. For much more Fibonacciana, see [8].

## 1.2 A Second Try

Here is a more successful approach to the question of what a typical $F_n$ looks like. Take one of the many codings of Fibonacci numbers and answer the question there. For example, $F_n$ counts the number of binary strings of length $n-2$ with no two consecutive ones:

$$F_5 = 5 \longleftrightarrow \{000, 001, 010, 100, 101\}$$

Let $\mathcal{F}_n$ be the Fibonacci strings of length $n$. So $|\mathcal{F}_n| = F_{n+2}$. This mismatch in notation is unfortunate, but keeping the classical notation for $F_n$ makes the literature easier to use.

For this coding, it is natural to ask, "What does a typical element of $\mathcal{F}_n$ look like?" Throughout, use the uniform distribution

$$P_n(x) = 1/F_{n+2}$$

on $\mathcal{F}_n$. This distribution is well known in statistical physics as the "hardcore model in 1-D." Let $X_i(x)$ be the $i$th bit of $x$. Natural questions are:

- What is the chance that $X_i = 1$?
- What is the distribution of $X_1 + \cdots + X_n$?
- How long is the longest zero run of $X_1, X_2, \ldots, X_n$?
- What is the waiting time distribution for the first one?

Indeed, any question asked for coin tossing can be asked for $\mathcal{F}_n$. In [5], a simple, efficient algorithm is given for exact generation of a uniformly chosen element of $\mathcal{F}_n$ (using the Fibonacci number system). Of course, there is literature on the mixing time of the natural Markov chain for generating from the uniform distribution on $\mathcal{F}_n$ – pick a coordinate at random; try to change to its opposite – see [10].

The main results developed (Propositions 1.1 to 1.6): as a process, $X_1, X_2, \ldots, X_n$ is close to a binary Markov chain $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$, where, with

$$\theta = \frac{\sqrt{5}-1}{2} \doteq 0.6180$$

so $\theta + \theta^2 = 1$,

$$P\{\tilde{X}_1 = 0\} = \theta, \qquad \text{with transition matrix } P = \begin{pmatrix} \theta & \theta^2 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

The closeness is in total variation for $X_1, \ldots, X_k$ with $k = n - f(n)$, $f(n) \to \infty$. This is strong enough to give useful answers to the previous four questions and many others. Let us turn now to mathematics.

**Proposition 1.1.** *For any $i$, $1 \le i \le n$,*

$$P_n(X_i = 0) = \frac{F_{i+1}F_{n+2-i}}{F_{n+2}}, \qquad P_n(X_i = 1) = \frac{F_i F_{n+1-i}}{F_{n+2}}.$$

*Proof.* Sequences with $X_i = 0$ may begin with any Fibonacci sequence of length $i-1$ ($F_{i+1}$ choices) and end with any Fibonacci sequence of length $n-i$ ($F_{n-i+2}$ choices). Dividing by the total number of Fibonacci sequences of length $n$ ($F_{n+2}$) gives the first result. The second is similar; a one in position $i$ forces zeros at $i-1$, $i+1$. After this, the start and end are arbitrary Fibonacci sequences. □

For subsequent use, recall that, with

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2} = 1-\phi = \frac{1}{-\phi} \doteq -0.6180,$$

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \tag{1.2}$$

and $F_n$ is the closest integer to $\phi^n/\sqrt{5}$. Recall $\theta = (\sqrt{5}-1)/2$. Combining (1.2) and Proposition 1.1, standard asymptotics gives

**Proposition 1.2.**

*(a)* $P_n(X_i = 0) = P_n(X_{n-i+1} = 0), \quad 1 \le i \le n.$ *(symmetry)*

*(b)* $P_n(X_1 = 0) = \theta\left[1 + O(\phi^{-2n})\right].$

*(c)* $P(X_i = 0) = \dfrac{\theta^2}{\sqrt{5}}\left[1 + O(\phi^{-2i}) + O(\phi^{-2(n-i)})\right].$ □

**Remark.** Part (c) shows, if $i$ and $n-i$ are large,

$$P(X_i = 0) \sim \frac{\theta^2}{\sqrt{5}}, \qquad P(X_i = 1) \sim 1 - \frac{\theta^2}{\sqrt{5}}.$$

This of course is the stationary distribution of the transition matrix in (1.2). It is useful to collect together properties of the Markov chain.

**Proposition 1.3.** *For* $\theta = (\sqrt{5}-1)/2$, *let a Markov chain* $\{\tilde{X}_n\}$ *on* $\{0, 1\}$ *have transition matrix*

$$P = \begin{pmatrix} \theta & \theta^2 \\ 1 & 0 \end{pmatrix},$$

*starting distribution* $P(\tilde{X}_1 = 0) = \theta$, $P(\tilde{X}_1 = 1) = \theta^2$. *Then* $P$ *has stationary distribution*

$$\pi(0) = \frac{\theta^2}{\sqrt{5}}, \qquad \pi(1) = 1 - \frac{\theta^2}{\sqrt{5}},$$

*and P is reversible. The eigenvalues are* $\beta_0 = 1$, $\beta_1 = \theta - 1$. *The right eigenvectors are*

$$f_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad f_1 = \begin{pmatrix} 1 \\ -1/\theta^2 \end{pmatrix}.$$

*If the chain is denoted* $\tilde{X}_i$, $1 \le i < \infty$, *for all n and* $e_1, \ldots, e_n \in \{0, 1\}$ *an allowable sequence,*

$$P(\tilde{X}_1 = e_1, \tilde{X}_2 = e_2, \ldots, \tilde{X}_n = e_n) = \theta^{n+e_n}. \qquad \square$$

The main result of this section gives an explicit bound between the probability distribution $\mu_{n,k}$ of $X_1, \ldots, X_k$ from the Fibonacci chain and $\tilde{\mu}_k$ the probability distribution of the Markov chain $\tilde{X}_1, \ldots, \tilde{X}_k$ as in Proposition 1.3. The total variation distance is

$$\|\mu_{nk} - \tilde{\mu}_k\|_{\mathrm{TV}} = \max_{A \subseteq C_2^k} |\mu_{n,k}(A) - \tilde{\mu}_k(A)|.$$

**Proposition 1.4.** *With notation as earlier, for all n and* $1 \le k \le n$,

$$\|\mu_{n,k} - \tilde{\mu}_k\| = O(\theta^{2(n-k)}).$$

*Proof.* As usual,

$$\|\mu_{nk} - \tilde{\mu}_k\|_{\mathrm{TV}} = \frac{1}{2} \sum_{x \in C_2^k} |P(X_1 = x_1, \ldots, X_k = x_k) \\ - P(\tilde{X}_1 = x_1, \ldots, \tilde{X}_k = x_k)|.$$

For any $x_1, \ldots, x_k$,

$$P_n(X_1 = x_1, \ldots, X_k = x_k) = \frac{F_{n+2-k-x_n}}{F_{n+2}},$$
$$P\{\tilde{X}_1 = x_1, \ldots, \tilde{X}_n = x_n\} = \theta^{k+x_i}.$$

It follows that

$$\|\mu_{nk} - \tilde{\mu}_k\|_{\mathrm{TV}} = \frac{1}{2} \left\{ \left| \frac{F_{n+2-k}}{F_{n+2}} - \theta^k \right| F_{k+1} + \left| \frac{F_{n+1-k}}{F_{n+2}} - \theta^{k+1} \right| F_k \right\}.$$

The claimed result now follows from (1.2) in a straightforward manner. $\square$

Some of the preceding questions have been previously answered. Let $S_n = X_1 + \cdots + X_n$. Diaconis, Graham, and Holmes [5] prove

**Proposition 1.5.**

$$(a)\ P_n(S_n = k) = \frac{\binom{n+1-k}{k}}{F_{n+2}}, \quad 0 \le k \le \left\lfloor \frac{n+1}{2} \right\rfloor.$$

$$(b)\ E_n(S_n) = (n+1)\frac{\sqrt{5}-1}{2\sqrt{5}} + \frac{1}{5\phi} + O(n\phi^{-2n}),\ \mathrm{Var}(S_n) = \frac{n+1}{5\sqrt{5}} + O(1).$$

$$(c)\ P_n\left(\frac{S_n - E_n}{\sqrt{\mathrm{Var}_n}} \le x\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad \text{as } n \text{ tends to infinity.} \qquad \square$$

The longest zero run can be determined by solving the problem for the Markov chain and then transferring it to $X_1, X_2, \ldots, X_n$ using Proposition 1.4.

**Proposition 1.6.** *Let $M_n$ be the longest zero run for a uniform element of $\mathcal{F}_n$. Then $M_n/\log_{1/\theta} n \to 1$ in probability.*

*Proof.* Proposition 1.6 follows by first proving the parallel result for the Markov chain $\tilde{X}_i$ and then tranferring to $X_i$ using Proposition 1.4. Let $l = l(n) = \lfloor \log_{1/\theta} n \rfloor$. From Proposition 1.3, the transition matrix $P$ is explicitly diagonalized as $P = VDV^{-1}$, with $V$ the matrix with column vectors the right eigenvectors of $P$, and $D$ a diagonal matrix of eigenvalues

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1/\theta^2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -\theta^2 \end{pmatrix}, \quad V^{-1} = \frac{\theta^2}{1+\theta^2} \begin{pmatrix} 1/\theta^2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Here $\theta + \theta^2 = 1$ and the stationary distribution is $\pi(0) = 1/(1+\theta^2)$, $\pi(1) = \theta^2/(1+\theta^2)$. Thus

$$P_0\{\tilde{X}_i = 0\} = P^i(0,0) = (VD^iV^{-1})_{00} = \pi(0) + O(\theta^{2i}),$$
$$P_1\{\tilde{X}_i = 0\} = \pi(0) + O(\theta^{2i}).$$

Since $P_a\{\tilde{X}_i = \tilde{X}_{i+1} = \cdots = \tilde{X}_{i+l-1} = 1\} = P_a\{\tilde{X}_i = 0\}P^{l-1}(0,0)$, for either starting state $a$, for any starting distribution $\sigma$,

$$P_\sigma(\tilde{X}_i = \cdots = \tilde{X}_{i+l-1} = 1) = \pi(0)P^{l-1}(0,0) + O\left(\theta^{2i}P^{l-1}(0,0)\right). \qquad (1.3)$$

From this,

$$P_\sigma\left\{\tilde{M}_n \ge (1+\epsilon)l\right\} = P_\sigma\left\{\bigcup_{i=0}^{n-l} [\text{0-run from } i \ge (1+\epsilon)l]\right\}$$
$$\le (n-l)\pi(0)P^{\lfloor l(1+\epsilon)\rfloor}(0,0) + O\left(P^{\lfloor l(1+\epsilon)\rfloor}(0,0)\right).$$

From the choice of $l$, $P^{\lfloor l(1+\epsilon)\rfloor}(0,0) = O(1/n^{1+\epsilon})$, so the right-hand side tends to 0.

To show that $\tilde{M}_n/l \geq (1-\epsilon)$ with high probability, split $[n]$ into disjoint blocks of length $\lfloor l(1-\epsilon)\rfloor$. Let $\tilde{Y}_i$ be 1 or 0 as the $i$th block is all 0's or not. Let $\widetilde{W} = \sum_{i=1}^{n/l(1-\epsilon)} \tilde{Y}_i$. The second moment method will be used to show $P\{\widetilde{W} > 0\} \to 1$. From (1.3), with $l$ replaced by $\lfloor l(1-\epsilon)\rfloor$,

$$E(\widetilde{W}) = \frac{n}{l(1-\epsilon)}\pi(0)P^{\lfloor l(1-\epsilon)\rfloor}(0,0) + O\left(P^{\lfloor l(1-\epsilon)\rfloor}(0,0)\right) \sim \frac{n^{\epsilon}\pi(0)}{l(1-\epsilon)} \tag{1.4}$$

$$\mathrm{Var}(\widetilde{W}) = \sum_i \mathrm{Var}(\tilde{Y}_i) + 2\sum_{i<j}\mathrm{Cov}(\tilde{Y}_i, \tilde{Y}_j). \tag{1.5}$$

Since $\tilde{Y}_i$ are binary, the asymptotics of the first sum in (1.5) are as in (1.4). Using the Markov property, $P_\sigma(\tilde{Y}_i = \tilde{Y}_j = 1) = P_\sigma(\tilde{Y}_i = 1)P_0(\tilde{Y}_{j-i} = 1)$. The terms may be bounded using (1.3) and the second sum is of order $n/n^{2(1-\epsilon)}$. It follows that the variance of $\widetilde{W}$ is of the same order as the mean, so a Chebyshev bound shows

$$P\left\{\frac{\tilde{M}_n}{l} > 1-\epsilon\right\} \longrightarrow 1.$$

This proves Proposition 1.6 with $M_n$ replaced by $\tilde{M}_n$. The transfer of the limit theorem back to $M_n$ is routine from Proposition 1.4. □

**Remark.** More refined limiting behavior of $M_n$ will surely be colored by the nonexistence of limiting behavior associated with the maximum of discrete random variables. See [3, 13].

I cannot leave this topic without remarking on some amazing formulas communicated to me by Richard Stanley. Throughout, let $X_i(x)$ be the $i$th binary digit of a uniformly chosen point in $\mathcal{F}_n$. Define a random variable

$$W_n(x) = \prod_{1\leq i\leq n} i^{X_i(x)}.$$

Stanley (in personal correspondence) shows

$$E(W_n) = \frac{1}{F_{n+2}}\sum_{\lambda\vdash n+1} f(\lambda), \qquad E(W_n^2) = \frac{1}{F_{n+2}}(n+1)!. \tag{1.6}$$

In (1.6), $f(\lambda)$ is the dimension of the irreducible representation of the symmetric group $S_n$ corresponding to the partition $\lambda$. It is well known [11, pp. 62–64] that $\sum f(\lambda)$ equals the number of involutions in $S_{n+1}$ and $\sum f^2(\lambda) = (n+1)!$.

The formulas (1.6) are sufficiently surprising that a numerical check seems called for. Consider $n = 3$; $S_4$ has 10 involutions and 24 elements. For $W_n(x)$ the product over the empty set is 1:

| $x$ | 000 | 100 | 010 | 001 | 101 | Sum |
|---|---|---|---|---|---|---|
| $W_3(x)$ | 1 | 1 | 2 | 3 | 3 | 10 |
| $W_3^2(x)$ | 1 | 1 | 4 | 9 | 9 | 24 |

The asymptotics of $\sum f(\lambda)$ are well known [11, p. 64]. This gives

$$E(W_n) \sim \frac{(n+1)^{(n+1)/2} \exp\left\{-(n+1)/2 + \sqrt{n+1} - 1/2\right\}}{\sqrt{2}\, F_{n+2}},$$

$$\mathrm{Var}(W_n) \sim \frac{(n+1)!}{F_{n+2}} \sim \frac{(n+1)^{(n+1)}\, e^{-(n+1)}}{\sqrt{2\pi n}\, F_{n+2}}.$$

From this, we see that $W_n$ is concentrated around its mean. Proposition 1.3 and easier calculations show that $L_n = \log(W_n)$ has a limiting normal distribution, so $W_n$ is log normal in the limit.

A somewhat contrived set of steps leading to consideration of $W_n$ may be constructed as follows. Suppose one wanted to consider a random square-free number with factors at most $x$. One natural way to do this considers the uniform distribution. Another natural approach is to consider $\epsilon_1, \ldots, \epsilon_n$ independent, 0/1 random variables with $P(\epsilon_i = 1) = P(\epsilon_i = 0) = 1/2$ and define

$$\widetilde{W}_n = \prod_{1 \le i \le n} p_i^{\epsilon_i}$$

with $2 = p_1 < p_2 < \cdots < p_n$ the distinct primes. An easier (but quite similar) problem considers

$$\widetilde{\widetilde{W}}_n = \prod_{1 \le i \le n} i^{\epsilon_i}.$$

This has

$$E\left(\widetilde{\widetilde{W}}_n\right) = \frac{n!}{2^n}, \qquad E\left(\widetilde{\widetilde{W}}_n^{\,2}\right) = \frac{(n!)^2}{2^n}.$$

So again $\widetilde{\widetilde{W}}_n$ is concentrated about its mean and $\log \widetilde{\widetilde{W}}_n$ is asymptotically normal. The random variable $W_n$ is the Fibonacci version of $\widetilde{\widetilde{W}}_n$. (Okay, okay; I said it was contrived.)

Stanley's motivation comes from his theory of differential posets. In [14, Problem 8], Stanley constructed a sequence of semisimple algebras $A_n$ of dimension $(n+1)!$ whose irreducible representations have degree $W_n(x)$ for $x \in \mathcal{F}_n$. Thus the number of irreducible representations is $F_{n+2}$. The existence of $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ (with nice restriction properties) is not hard to show by "general nonsense." A useful set of generators and relations was found by

Okada [12]. This has spawned a host of interesting developments that may be found by following the citations to Okada's paper.

The tension between recreational math and "real math" is evident throughout the Fibonacci world. As a parting shot, I offer the following: 144 is a Fibonacci number and it's also a perfect square (uh-oh). Also, 8 is a Fibonacci number that is a cube (uh-oh). Are there any others? No! Bugeaud et al. [1] proved that 1, 8, 144 are the only Fibonacci numbers that are powers. Their proof makes real use of the full machine of modern number theory.

There are other codings of $F_n$; see https://oeis.org/A000045 at the On-Line Encyclopedia of Integer Sequences. Also, in [15], parts (b), (c), and (d) of Exercise 1.35 are about compositions with specified parts. In [16], part (a) of Exercise 7.66 has a cute proof. There are also Lucas numbers; I don't know any codings for them. Some of these suggest fresh questions. Fortunately, I can call Ron.

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