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Introduction

Differential topology, like differential geometry, is the study of smooth (or 'differential') manifolds. There are several equivalent versions of the definition: a common one is the existence of local charts mapping open sets in the manifold M^m to open sets in $\mathbb{R}^m$, with the requirement that coordinate changes are smooth, i.e. infinitely differentiable.

If M and N are smooth manifolds, a map $f : M \to N$ is called smooth if its expressions by the local coordinate systems are smooth. This leads to the concept of smooth embedding. If $f : M \to N$ and $g : N \to M$ are smooth and inverse to each other, they are called diffeomorphisms: we can then regard M and N as copies of the same manifold. If f and g are merely continuous and inverse to each other, they are homeomorphisms. Thus homeomorphism is a cruder means of classification than diffeomorphism.

The notion of smooth manifold gains in concreteness from the theorem of Whitney that any smooth manifold M^m may be embedded smoothly in Euclidean space $\mathbb{R}^n$ for any $n \geq 2m + 1$, and so may be regarded as a smooth submanifold of $\mathbb{R}^n$, locally defined by the vanishing of $(n - m)$ smooth functions with linearly independent differentials. An important example is the unit sphere S^{n-1} in $\mathbb{R}^n$. The disc D^n bounded by S^{n-1} is an example of the slightly more general notion of manifold with boundary.

Whitney's result is more precise: it states that (if M is compact) embeddings are dense in the space of all maps $f : M^m \to \mathbb{R}^n$, suitably topologised, provided $n \geq 2m + 1$, and more generally the same holds for maps $M^m \to N^n$ for any manifold N of dimension n. Other 'general position' results include the fact that if $m > p + q$, a map $f : P^p \to M^m$ will in general avoid any union of submanifolds of M of dimension $\leq q$. These results can be deduced from the general transversality theorem, which also applies to permit detailed study of the local forms of singularities of smooth maps.

One of the ultimate aims of differential topology is the classification up to diffeomorphism of (say, compact) smooth manifolds, and while this is algorithmically impossible in dimensions ≥ 4 on account of the corresponding result for finitely presented groups, we can perform it in some cases of interest. The technique is to reduce first to a problem in homotopy theory, and solve that using algebraic techniques. A basic requirement is a reasonably intrinsic way to describe manifolds: this is provided by a handle presentation.

Another central question is the possibility or otherwise of finding an embedding of a given manifold V^v in a given manifold M^m of larger dimension. Whitney himself found a key technique in the first tricky case $m = 2v$, and his idea was extended to general results in a range, roughly $m > \frac{3}{2}v$.

Classification results are accompanied by theorems giving methods of constructing manifolds: here we prescribe the homotopy type (which must satisfy Poincaré duality) and further 'normal' structure, apply transversality to construct something, and then endeavour to perform surgery to obtain the desired result.

Classification up to diffeomorphism is very fine, and only available in a few cases. The equivalence relation given by cobordism is much cruder, but is generally applicable and computable. Extensive calculations are available, and indeed through these, differential topology feeds back as a tool in pure homotopy theory.

Although the foundations have much in common with differential geometry, we approached the subject from a background in algebraic topology, and this book is written from that viewpoint. The study of differential topology stands between algebraic geometry and combinatorial topology. Like algebraic geometry, it allows the use of algebra in making local calculations, but it lacks rigidity: we can make a perturbation near a point without affecting what happens far away. While the classification results are close to those for combinatorial manifolds, the differential structure gives access to a rich source of techniques.

While the notion of differentiable manifold had gradually evolved over a century, differential topology as a subject was to a large extent begun by Whitney, with a major paper [175] in 1936 which, as well as clarifying the notion of 'differentiable manifold', established several foundational results. He obtained further important results in [176] and [177]. Spectacular new ideas were introduced in 1954 by Thom [150] on cobordism and in 1956 by Milnor [92] on differential structures on S^7. From then on, the pace of development was rapid, with contributions by numerous mathematicians. The author personally was inspired by lectures and writings of Milnor.

In a somewhat separate development, there was great progress in studying group actions. The solution of Hilbert's fifth problem [106], while independent of the study of smooth actions, gave impetus to the whole area. Major results were established by Montgomery, Mostow, and others in papers (for example, [104], [111], [112], [105]) in the 1950s. The publication of the seminar notes [20] was a landmark. The paper [119] extended key results to the case of proper actions. By 1960 this topic had been absorbed in the mainstream of geometric topology.

Many of the central problems in the topology of manifolds had been solved (or reduced to problems in homotopy theory) by 1970: in §7.8 we describe how to approach diffeomorphism classification and give some examples, and in Theorem 6.4.8 we give a result dealing with smooth embeddings. As a result, the focus of current research gradually shifted elsewhere.

The original draft of this book was written at a time when differential topology was new and exciting, and there were no books on the topic. While there now exist introductory accounts and books on particular areas of differential topology, there does not seem to be any other that does justice to the breadth of the subject.

This book falls roughly in two halves: introductory chapters with general techniques, then four chapters, each including a major result. There are also two appendices.

We begin in Chapter 1 with the definitions of smooth manifold, manifold with boundary, and tangent bundle. We give equivalent formulations of the definition, and go on to techniques for piecing together local constructions, which are fundamental for much that follows.

It is often convenient to regard a manifold as formed by fitting pieces together, and we deal with several aspects of this process in Chapter 2. We introduce and establish the main results about tubular neighbourhoods, which form the main pieces. We give the necessary details about cutting and glueing, including a discussion of corners and how to straighten them.

Chapter 3 opens with basic definitions of Lie groups and of group actions, and some basic properties. The key to the geometric description of the actions is the notion of slice. The existence of slices was established in [104] for actions of compact groups, and was extended to proper group actions by Palais [119]. Slices lead to local models for actions, which allow us to extend many of the results of the first chapters of this book to the case of group actions: leading notably to existence of invariant metrics and (with some necessary restrictions) equivariant embeddings in Euclidean space. We go on to define and study the stratification of a proper G-manifold by orbit types, which to some extent reduces the classification problem for actions to problems not involving the group, and illustrate by discussing the case when there are at most two orbit types.

In Chapter 4 we treat 'general position' arguments, which are of frequent use in constructions. We can begin with the naive idea that one can push a k-dimensional subset and a v-dimensional one apart in a manifold of dimension $n > k + v$. However – and this is one area where differential topology is much richer than piecewise linear or 'pure' topology – we can apply the same basic idea at the level of jets to study singularities. The key underlying result is the transversality theorem. This whole subject has developed enormously, particularly after the work of John Mather in the sequence of papers [88]. We have tried to steer a middle course, keeping to fairly direct arguments, obtaining details on the results wanted elsewhere in this book, and giving a brief introduction to the study of singularities.

If M is a manifold with boundary ∂M and $f : S^{r-1} \times D^{m-r} \to \partial M$ an embedding, the union $M \cup_f h^r$ of M and $D^r \times D^{m-r}$, with the copies of $S^{r-1} \times D^{m-r}$ identified by f, is said to be obtained from M by attaching an r-handle (some care is necessary at the corner $S^{r-1} \times S^{m-r-1}$). A handle can be studied via the embedding of the sphere $f \mid (S^{r-1} \times \{0\})$, and extending to a tubular neighbourhood. Any compact manifold admits a decomposition into finitely many handles. In Chapter 5 we develop handle theory up to the central result, the h-cobordism theorem. Here we have taken the approach of forming a manifold by glueing pieces together, rather than manipulating a function on a fixed manifold: the latter is in some ways more elegant, but the former seems more perspicuous. The h-cobordism theorem is the key result enabling classification of manifolds up to diffeomorphism, and we illustrate with a few examples of explicit diffeomorphism classifications. The absence of any such result for 4-manifolds means that no such classifications exist here. In the detailed treatment we restrict to the simply connected case, but describe briefly in a final section how to modify the theorem for the general case.

In Chapter 6, on immersions and embeddings, we include an account of Smale–Hirsch theory, which gives a reduction of the classification of immersions to a homotopy problem. We then describe in full Whitney's method of removing self-intersections of an n-manifold in a $(2n)$-manifold, and Haefliger's extension of the method to obtain a full theory of embeddings and immersions in the metastable range, giving (when they apply) necessary and sufficient conditions for a given map to be homotopic to an embedding (or immersion) or for two homotopic embeddings to be diffeotopic.

Next, Chapter 7 gives a full account of the theory of surgery (in the simply connected case), with a number of applications. This restriction allows a much simpler presentation than in my book [167], closer to the original papers, but the approach is the same. Sections are included on the relevant pieces of quadratic algebra, and on Poincaré complexes and maps of degree 1. A section on homotopy theory of Poincaré complexes includes a discussion of Spivak's theorem and its uses, and a brief account of Brown's treatment of the Kervaire invariant.

Finally, in Chapter 8, we tackle the topic of cobordism, describe the main geometrical ideas, and show how to build up cobordism groups, rings, and bordism as a homology theory. We also give accounts of calculations of unoriented, unitary, oriented, and (perhaps rather ambitiously) special unitary bordism. Here we suppress many details which would require an extensive knowledge of homotopy theory; even so, much more is demanded of the reader than in earlier chapters. A final section ties together much of the preceding with an account of homotopy spheres and their embedding in the standard sphere.

Each chapter opens with a summary of its contents and concludes with a 'Notes' section consisting of historical remarks, key references, and notes on additional developments.

There are two appendices. Appendix A opens in §A.1 and §A.2 with a summary of useful results from analytic topology; §A.3 gives the results we need about proper group actions; and §A.4 offers a treatment of the requisite results on the topology of mapping spaces.

I attempt a bird's eye view of homotopy theory in Appendix B: here I aim to include the necessary definitions with (I hope) enough connecting material to make them intelligible, but cannot attempt a full exposition. In §B.1, I give basic terminology and describe the general framework for homotopy notions. The next section §B.2 gives definitions and basic properties of (mostly classical) Lie groups and classifying spaces. In §B.3, I list a number of calculations including those to which reference is made in the main text. Finally, §B.4 gives very brief introductions to skeletons, connected covers, Eilenberg–MacLane spaces and cohomology operations, and spectra.

The focus of this book is on the geometric techniques required for the study of the topology of smooth manifolds. One important tool I do not use is the calculus of differential forms, and its application (de Rham) to calculating real cohomology: for an account in this spirit I refer to the book [23] by Bott and Tu. I also eschew the technical details required for the comparisons of different kinds of structure: differential vs. combinatorial, C^∞ vs. C^r or vs. real analytic: these questions are not considered here, though I do pay some attention to comparing smooth and topological structures. Although I introduce, and use, Riemannian metrics, I am not concerned with properties of the metric, so feel free to choose a convenient metric when required. Symplectic structures are outside the scope of this book: the methods devised in the last 30 years for their study are of a different nature to those studied here.

In keeping with the original setting, I assume elementary analysis, but quote (with references) some results from analysis that are needed. I will take basic topological ideas and results as understood (there is a very brief account in §A.1).

I also assume a certain background in algebraic topology, though this is not needed in Chapters 1–4, and in Chapter 5 only basic homology theory is used. All chapters are to a large extent independent (in particular, Chapter 6 is independent of Chapter 5, so the forward references do not give problems); in general they are ordered so that later chapters use an increasing knowledge of homotopy theory.

The first draft of most of the book (Chapters 1–5, 7) was a series of duplicated notes based on seminars in the early 1960s. Chapters 1, 2, and 4 were originally based on a seminar held in Cambridge 1960–61. For the original notes, it seemed desirable to elaborate the foundations considerably beyond the point from which the lectures started, and the notes expanded accordingly. For these, I am indebted to all the Cambridge topology research students of the time for participating in the seminar, in particular to P. Baxandall, and to Steve Gersten for considerable assistance in writing up. For Chapters 1 and 2 this book remains fairly close to the original notes. However for Chapter 4, the area has developed enormously in the interim, particularly after Mather's work. So I have rewritten most; in doing so I have tried to steer a middle course, keeping to fairly direct arguments, but obtaining details on the results wanted elsewhere.

The original notes for Chapter 3 were issued a few years later, with thanks to Peter Whitham. They were an attempt to pull together results from several sources to get a coherent theory. The main source was the volume [20]. This focussed on topological (rather than smooth) actions; indeed its first section was on homology manifolds. Thus much of the emphasis on my notes was also on questions of analytic topology. The account presented in this chapter is thus completely rewritten: not only does it go well beyond the content of my old notes, but has a very different emphasis.

Chapter 5 on handle decompositions, leading up to the h-cobordism theorem, is based on lectures given and seminars held in Oxford (1962) and Cambridge (1964). Thanks are due to numbers of the then research students for their participation; in particular to the late Charles Thomas and to Denis Barden. I am indebted to Shu Otsuka for rendering the original notes of these chapters into LaTeX, and to Iain Rendall for drawing the diagrams.

The remainder of the book has been newly written. Chapter 6 follows the plan I had formed back in the 1960s. Chapter 7, although much simpler in detail, was informed by the same philosophy as my book [167]. The first part of Chapter 8 is based on my old seminar notes 'Cobordism: geometric theory' issued in Liverpool about 1965, but I felt that to give the chapter substance it was also necessary to include some significant calculations.

Thanks are due to Andrew Ranicki for encouraging me to turn the old notes into a book: a time-consuming, but agreeable task.

1

Foundations

If we start from the notions of curve – of dimension 1, locally like the line $\mathbb{R}$ of real numbers, and surface – of dimension 2, locally like the plane $\mathbb{R}^2$, the general term is ‘manifold’. We begin with perhaps the most elegant form of the definition, but will prove it equivalent to other versions.

We say that a function F defined on $\mathbb{R}^n$ (or on an open subset thereof) is *smooth* if it admits continuous partial derivatives of *all* orders. We use the term ‘smooth’ in this sense throughout.

In the opening section, we begin with the definition of smooth manifold, introduce the bump function, and proceed to the construction of partitions of unity. We then discuss connectedness.

Probably the most important property distinguishing smooth from topological manifolds is the existence of tangent vectors. Again we begin with a formal definition, then give alternative ways to view the concept. We introduce smooth maps, and discuss concepts of submanifold and embedding.

The tangent vectors to a smooth manifold form a vector bundle, so we next introduce the notions of Lie group and of fibre bundle, and establish the existence of a Riemannian structure on any smooth manifold.

An essential tool in the study of smooth manifolds is the integration of smooth vector fields. This becomes effective when combined with the use of partitions of unity to construct vector fields. We show how to reformulate the basic theorem asserting the existence solutions of ordinary differential equations in geometrical terms to yield flows on smooth manifolds.

Finally we extend the concept of smooth manifold to that of manifold with boundary, and establish the existence of a collar neighbourhood of the boundary.

1.1 Smooth manifolds

A *smooth m-manifold* is a Hausdorff topological space M^m with a family $\mathcal{F} = \mathcal{F}_M$ of continuous real-valued functions defined on M and satisfying the following conditions:

(M1) $\mathcal{F}$ is local. If $f : M \to \mathbb{R}$ is such that each point of M has a neighbourhood in which f agrees with a function of $\mathcal{F}$, then $f \in \mathcal{F}$.

(M2) $\mathcal{F}$ is differentiably closed. If $f_1, \dots, f_k \in \mathcal{F}$, and F is a smooth function on $\mathbb{R}^n$, then $F(f_1, \dots, f_k) \in \mathcal{F}$.

(M3) $(M, \mathcal{F})$ is locally Euclidean. For each point $P \in M$, there are m functions $f_1, \dots, f_m \in \mathcal{F}$ such that $Q \mapsto (f_1(Q), \dots, f_m(Q))$ gives a homeomorphism of a neighbourhood U of P in M onto an open subset V of $\mathbb{R}^m$. Every function $f \in \mathcal{F}$ coincides on U with $F(f_1, \dots, f_m)$, where F is a smooth function on V.

(M4) M is a countable union of compact subsets.

We call functions $f \in \mathcal{F}$ *smooth functions* of M, and the mapping defined in (M3) (or, by abuse of language, the set U) a *coordinate neighbourhood* of P. It follows from (M2) that sums, products, and constant multiples of smooth functions are also smooth.

The integer m is called the *dimension* of the manifold M.

We now give some simple examples of smooth manifolds.

The empty set is a smooth m-manifold (the definition is vacuously satisfied).

Euclidean space $\mathbb{R}^m$, with smooth functions taken in the ordinary sense, is a smooth m-manifold. Condition (M1) is trivial; (M2) follows from the rule for differentiating a composite (a function of a function); for (M3), since the coordinate functions are smooth, we take the identity map; and $\mathbb{R}^m$ is the union of the compact subsets given by $\|x\| \leq n$.

The disjoint union of a finite or countable set of smooth m-manifolds is another. Define a function to be smooth if the induced function on each part is so; the conditions are then all trivial.

Let O be an open subset of $\mathbb{R}^m$. Write $\mathcal{G}_O$ for the restriction to O of functions of $\mathcal{F}_{\mathbb{R}^m}$; $\mathcal{F}_O$ for the set of functions locally agreeing with a function of $\mathcal{G}_O$. Then since O is open in $\mathbb{R}^m$, $(O, \mathcal{G}_O)$ satisfies conditions (M1), (M3); $(O, \mathcal{F}_O)$ satisfies them and also condition (M2).

For each positive integer i, consider the sets $D^m_x(\sqrt{m}/i)$[1] such that all the coordinates of ix are integers and which are contained in O. There are only countably many of these. For any $y \in O$, some $\mathring{D}^m_y(\delta) \subset O$. Choose $i > 2\sqrt{m}/\delta$. Then some x with $ix \in \mathbb{Z}^m$ is within a distance $\sqrt{m}/i$ of y, and

$$y \in D^m_x(\sqrt{m}/i) \subset D^m_y(2\sqrt{m}/i) \subset \mathring{D}^m_y(\delta) \subset O.$$

[1] For this notation and others, see the Index of Notations on p. 340.

Thus the chosen sets cover M, so (M4) also holds, and O is a smooth manifold.

More generally, let M be any smooth m-manifold and O be an open subset of M. Again write $\mathcal{G}_O$ for the restriction to O of functions of $\mathcal{F}_M$; $\mathcal{F}_O$ for the set of functions locally agreeing with a function of $\mathcal{G}_O$. We see as above that (M1)-(M3) hold. Now M is covered by coordinate charts, so any compact subset is covered by finitely many; hence M is covered by countably many charts U_α. Thus O is the union of countably many sets $O \cap U_\alpha$, each of which can be regarded as an open set in Euclidean space, so by the preceding paragraph is a countable union of compact sets. Thus (M4) also holds, and the structure of smooth m-manifold on M induces such a structure on O. We call O an *open submanifold* of M.

Let $M_1^{m_1}$, $M_2^{m_2}$ be smooth manifolds. Then the topological product $N^{m_1+m_2} = M_1^{m_1} \times M_2^{m_2}$ has a natural structure of smooth manifold. For let π_1, π_2 denote projections on the factors. Then for $f_1 \in \mathcal{F}_{M_1}$, $f_2 \in \mathcal{F}_{M_2}$, we define $f_1 \circ \pi_1$, $f_2 \circ \pi_2$ to belong to $\mathcal{F}_N$; any smooth functions of a finite set of these; and any function locally agreeing with one of these functions. This definition ensures that conditions (M1) and (M2) are satisfied. But so is (M3), for it now follows that if $\varphi_1 : U_1 \to \mathbb{R}^{m_1}$, $\varphi_2 : U_2 \to \mathbb{R}^{m_2}$ are coordinate neighbourhoods in M_1 and M_2, then $\varphi_1 \times \varphi_2 : U_1 \times U_2 \to \mathbb{R}^{m_1+m_2}$ can be taken as a coordinate neighbourhood in $M_1 \times M_2$. And (M4) follows since (see §A.2) the product of two compact sets is compact.

The first tool for working with our definition is a bump function. Define first a function B_1 on $\mathbb{R}$ by:

$$B_1(x) = \begin{cases} \exp\left(\frac{1}{(x(x-1))}\right) & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then B_1 is smooth, non-negative, and differs from zero when $0 < x < 1$. The *bump function* $Bp(x)$ is now given by

$$Bp(x) = \int_0^x B_1(t)dt \bigg/ \int_0^1 B_1(t)dt.$$

Since $B_1(x)$ is smooth, so is $Bp(x)$. Also

$$\begin{aligned} Bp(x) &= 0 \quad \text{if} \quad x \le 0, \\ 0 < Bp(x) &< 1 \quad \text{if} \quad 0 < x < 1, \quad \text{and} \\ Bp(x) &= 1 \quad \text{if} \quad x \ge 1. \end{aligned}$$

The bump function is illustrated in Figure 1.1. Although we have given an explicit construction, the above are the essential properties of the bump