LECTURES ON K3 SURFACES

K3 surfaces are central objects in modern algebraic geometry. This book examines this important class of Calabi–Yau manifolds from various perspectives in eighteen self-contained chapters. It starts with the basics and guides the reader to recent breakthroughs, such as the proof of the Tate conjecture for K3 surfaces and structural results on Chow groups. Powerful general techniques are introduced to study the many facets of K3 surfaces, including arithmetic, homological, and differential geometric aspects. In this context, the book covers Hodge structures, moduli spaces, periods, derived categories, birational techniques, Chow rings, and deformation theory. Famous open conjectures, for example the conjectures of Calabi, Weil, and Artin–Tate, are discussed in general and for K3 surfaces in particular, and each chapter ends with questions and open problems. Based on lectures at the advanced graduate level, this book is suitable for courses and as a reference for researchers.

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Lectures on K3 Surfaces

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Contents

Preface

This book originates from a graduate course in algebraic geometry held in the summer of 2010. I introduced many fundamental techniques in algebraic geometry and explained in detail how they are applied to K3 surfaces. The diversity of the theory of K3 surfaces, touching upon so many topics in both algebraic geometry and other areas, including arithmetic, complex and differential geometry, homological algebra, and even mathematical physics, is fascinating. I hoped to convey some of this fascination and, at the same time, to demonstrate how the various aspects – ranging from Hodge theory to moduli spaces to derived categories – come together in a meaningful way when studied for K3 surfaces.

Over time, the original lecture notes have grown. They now cover large parts, but by no means all, of the theory of K3 surfaces. As the notes made available online appeared to be useful, it seemed worthwhile to turn them into this book. I hope it will serve as an introduction to the subject as well as a guide to the vast literature. The balance between these two goals turned out to be difficult to achieve. Some chapters are more or less self-contained, while others are certainly not and rather are meant as an invitation to consult the original sources.

Each chapter is devoted to a different topic and presents the relevant theory in a condensed form, accompanied by extensive references to the original articles and to the relevant textbooks. Sometimes the text can be read as a survey, while other times technical aspects particular to K3 surfaces are discussed in detail. Often, I try to give ad hoc arguments that work only for K3 surfaces, though a more powerful general theory may be available. The aim is to allow the chapters to be read independently of each other, encouraging a non-linear reading. Although this goal was not fully achieved, I hope I at least made it easy to navigate between the chapters. Also, I have not hesitated to revisit some aspects to emphasize different angles or give more details.

The choice of topics is a personal one and I am aware of many omissions. Also, I have deliberately tried to avoid overlap with the existing accounts of larger parts of the theory as in the book [33] by Barth et al. or in the seminar notes [54] by Beauville et al., both of which are excellent introductions.

Important topics that are not covered here include the role of K3 surfaces in Claire Voisin's approach to Green's syzygy conjecture [49], the dynamical aspects of K3 surfaces as studied by Curt McMullen [403], Viatcheslav Kharlamov's results on real K3 surfaces [293], the work of Davesh Maulik, Rahul Pandharipande, and Richard Thomas on curve and sheaf counting on K3 surfaces in connection with Gromov–Witten and Donaldson–Thomas theory [399], questions related to rational points of K3 surfaces defined over number fields and the Brauer–Manin obstruction, K3 surfaces and conformal field theory [25], and many others. What is actually covered here is revealed by a quick look at the table of contents.

Prerequisites: As an advanced course, familiarity with the basic notions of algebraic geometry (schemes, varieties, cohomology of coherent sheaves, curves, and surfaces) and complex geometry (Kähler manifolds, Hodge theory) will be helpful.

Cross-references and proofs: Cross-references of the type 'Theorem **1**.2.3' refer to Theorem 2.3 in another chapter, here in Chapter **1**, whereas 'Proposition 2.3' refers to Proposition 2.3 within the same chapter.

If a proposition or a theorem concludes with a □, then either the arguments or the main ideas of the proof have been given earlier. If there is neither a proof nor a □, then the result needs to be looked up in the literature. Sometimes a proof is more like a sketch of the main ideas that should, however, allow one to reconstruct most of the details.

Acknowledgements: The two classics [33, 54] taught me much about the fundamental concepts in the theory of K3 surfaces, and discussions with François Charles and Davesh Maulik over recent years have further shaped my way of thinking about K3 surfaces. I gratefully acknowledge their influence.

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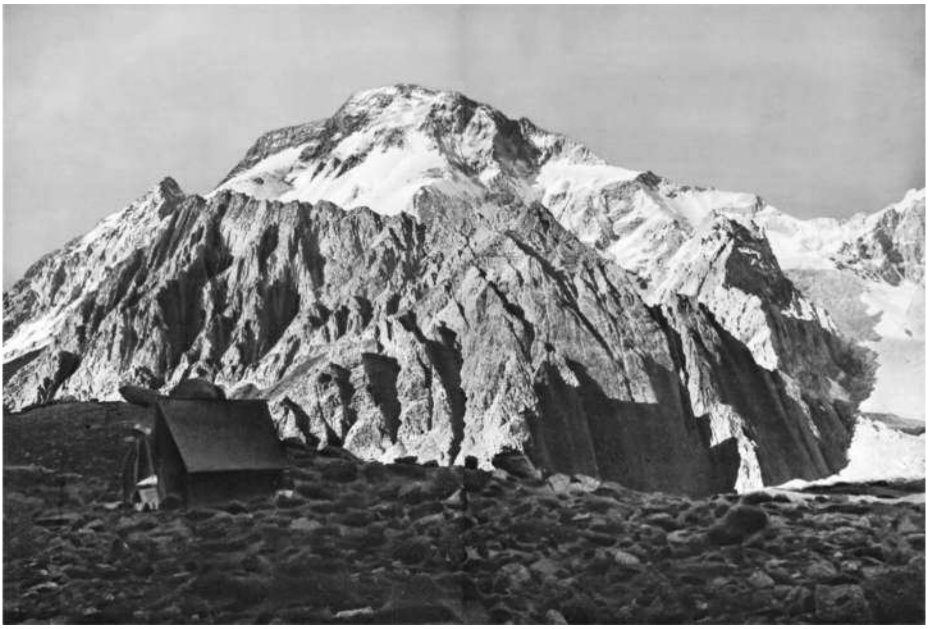

K3 at sunset, photographed by Vittorio Sella in 1909.[1]

The name 'K3 surface' was introduced by André Weil. In the comments to his 'Final report on contract AF 18(603)-57', written in 1958 and published in his collected papers [634], he writes:

> *il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.*

In fact, a mountain K3 exists as well. It is also known as Broad Peak and at 8051 m it is the twelfth highest mountain in the world and the fourth highest in Pakistan. Incidentally its first ascent took place in June 1957, which was around the time when Weil thought about K3 surfaces, and the first winter ascent occurred only in March 2013.

The first time the name 'K3 surface' appeared in a published article seems to be in Kodaira's [308]. Weil himself acknowledges discussions about K3 surfaces with Kodaira and Spencer during his visits to Princeton in 1958 and gives credit to Nirenberg, Andreotti, and Atiyah (whom he met during his stay in Cambridge in 1953–1954). Andreotti has never published his results on the local Torelli theorem for (Kummer) K3 surfaces, or any other result on K3 surfaces, but Grauert refers to them in [215]. In his paper [26] from 1958, Atiyah shows that Kummer surfaces and quartics are deformation equivalent, but general K3 surfaces are not discussed nor is the name mentioned.

[1] From Il Principe Luigi Amadeo di Savoia, Duca degli Abruzzi, *La Spedizione nel Karakoram e nell'Imalaia occidentale 1909,* relazione del Dott. Filippo de Filippi, illustr. da Vittorio Sella (Bologna: Zanichelli, 1912).