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1

Basic Definitions

Algebraic K3 surfaces can be defined over arbitrary fields. Over the field of complex numbers a more general notion exists that includes non-algebraic K3 surfaces. In Section 1, the algebraic variant is introduced and some of the most important explicit examples are discussed. Classical numerical invariants are computed in Section 2. In Section 3, complex K3 surfaces are defined and Section 4 contains more examples which are used for illustration in later chapters.

1 Algebraic K3 Surfaces

Let k be an arbitrary field. A *variety over k* (usually) means a separated, geometrically integral scheme of finite type over k.

Definition 1.1 A *K3 surface* over k is a complete non-singular variety X of dimension two such that[1]

$$\Omega^2_{X/k} \simeq \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

Once the base field is fixed, we often simply write Ω_X instead of $\Omega_{X/k}$. The canonical bundle of a non-singular variety X, i.e. the determinant of Ω_X, shall be denoted K_X or ω_X, depending on whether we regard it as a divisor or as an invertible sheaf.

By definition, the cotangent sheaf Ω_X of a K3 surface X is locally free of rank two and $\omega_X \simeq \mathcal{O}_X$. Moreover, the natural alternating pairing

$$\Omega_X \times \Omega_X \longrightarrow \omega_X \simeq \mathcal{O}_X,$$

[1] By definition, a variety over a field k is complete if the given morphism $X \longrightarrow \operatorname{Spec}(k)$ is proper and X over k is non-singular if the cotangent sheaf $\Omega_{X/k}$ is locally free of rank $\dim(X)$, which is equivalent to $X_{\bar{k}} := X \times_k \bar{k}$ being regular; see e.g. [375, Prop. 6.2.2].

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of which we think as an algebraic symplectic structure, induces a non-canonical isomorphism

$$\mathcal{T}_X := \Omega_X^* := \mathcal{H}om(\Omega_X, \mathcal{O}_X) \simeq \Omega_X.$$

Remark 1.2 Any smooth complete surface is projective. So, with the above definition, K3 surfaces are always projective.

There are various proofs for the general fact. For example, Goodman (see [235]) shows that the complement of any non-empty open affine subset is the support of an ample divisor. The proof in [33], written for smooth compact complex surfaces, uses fibrations of the surface associated with some rational functions. See [375, Ch. 9.3] for a proof over an arbitrary field.

Example 1.3 (i) A smooth *quartic* $X \subset \mathbb{P}^3$ is a K3 surface. Indeed, from the short exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

on $\mathbb{P}^3$ and the vanishings $H^1(\mathbb{P}^3, \mathcal{O}) = H^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0$ one deduces $H^1(X, \mathcal{O}_X) = 0$. Taking determinants of the conormal bundle sequence (see [236, II.Prop. 8.12])

$$0 \longrightarrow \mathcal{O}(-4)|_X \longrightarrow \Omega_{\mathbb{P}^3}|_X \longrightarrow \Omega_X \longrightarrow 0$$

yields the adjunction formula $\omega_X \simeq \omega_{\mathbb{P}^3} \otimes \mathcal{O}(4)|_X \simeq \mathcal{O}_X$. In local homogeneous coordinates with X given as the zero set of a quartic polynomial f, a trivializing section of ω_X can be written explicitly as the residue

$$\operatorname{Res}\left(\frac{\sum(-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_3}{f}\right), \tag{1.1}$$

which, for example, on the affine chart $x_0 = 1$ with affine coordinates y_1, y_2, y_3 is

$$\operatorname{Res}\left(\frac{dy_1 \wedge dy_2 \wedge dy_3}{f(1, y_1, y_2, y_3)}\right). \tag{1.2}$$

A particularly interesting special case is provided by the *Fermat quartic* $X \subset \mathbb{P}^3$ defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

In order for it to be smooth one has to assume $\mathrm{char}(k) \neq 2$.

(ii) Similarly, a smooth complete intersection of type $(d_1, \ldots, d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum d_i = n + 3$. Note that under the natural assumption that all $d_i > 1$ there are in fact only three cases (up to permutation): $n = 1, d_1 = 4$ (as in (i)); $n = 2, d_1 = 2, d_2 = 3$; and $n = 3, d_1 = d_2 = d_3 = 2$. This yields examples of K3 surfaces of degree four, six, and eight.

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(iii) Let k be a field of $\mathrm{char}(k) \neq 2$ and let A be an abelian surface over k.[2] The natural involution $\iota\colon A \longmapsto A$, $x \longmapsto -x$, has the 16 two-torsion points as fixed points. (They are geometric points and not necessarily k-rational.) The minimal resolution $X \longrightarrow A/\iota$ of the quotient, which has only rational double point singularities (cf. Section **14**.0.3), defines a K3 surface. K3 surfaces of this type are called *Kummer surfaces*. For details in the case of $k = \mathbb{C}$; see [43, Prop. VIII.11] and for a completely algebraic discussion [28, Thm. 10.6].[3]

An alternative way of describing X starts with blowing up the fixed points $\widetilde{A} \longrightarrow A$. Since the fixed points are ι-invariant, the involution ι lifts to an involution $\tilde{\iota}$ of $\widetilde{A}$. The quotient $\widetilde{A} \longrightarrow X$ by ι is a ramified double covering of degree two. A local calculation shows that smoothness of X and $\widetilde{A}$ are equivalent (in characteristic $\neq 2$).

$$\begin{array}{ccc} \tilde{\iota} \circlearrowright \widetilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ X & \longrightarrow & A/\iota. \end{array}$$

Moreover, the canonical bundle formulae for the blow-up $\widetilde{A} \longrightarrow A$ (cf. [236, V. Prop. 3.3]) and for the branched covering $\pi : \widetilde{A} \longrightarrow X$ (cf. [33, I.16] or [442, Ch. 6]) yield

$$\omega_{\widetilde{A}} \simeq \mathcal{O}(\Sigma E_i) \text{ and } \omega_{\widetilde{A}} \simeq \pi^*\omega_X \otimes \mathcal{O}(\Sigma E_i).$$

This shows $\pi^*\omega_X \simeq \mathcal{O}_{\widetilde{A}}$. Here, the E_i are the exceptional divisors of $\widetilde{A} \longrightarrow A$. Their images $\overline{E}_i$ in X satisfy $\pi^*\mathcal{O}(\overline{E}_i) \simeq \mathcal{O}(2E_i)$. Note that $\pi_*\mathcal{O}_{\widetilde{A}} \simeq \mathcal{O}_X \oplus L^*$, where the line bundle L is a square root of $\mathcal{O}(\sum \overline{E}_i)$, and hence $\pi^*\omega_X \simeq \mathcal{O}_{\widetilde{A}}$ implies $\omega_X \simeq \mathcal{O}_X$. Finally note that the image of the injection $H^1(X, \mathcal{O}_X) \hookrightarrow H^1(\widetilde{A}, \mathcal{O}_{\widetilde{A}}) = H^1(A, \mathcal{O}_A)$ is contained in the invariant part of the action induced by ι. Hence, $H^1(X, \mathcal{O}_X) = 0$. See Remark **14**.3.16 for a converse describing which K3 surfaces are Kummer surfaces.

The Fermat surface in (i) is in fact a Kummer surface, but this is not obvious to see; cf. Example **14**.3.18.

(iv) Consider a double covering

$$\pi : X \longrightarrow \mathbb{P}^2$$

branched along a curve $C \subset \mathbb{P}^2$ of degree six. Then $\pi_*\mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3)$ which in particular shows $H^1(X, \mathcal{O}_X) = 0$. Note that for $\mathrm{char}(k) \neq 2$ the surface X is non-singular if C is. The canonical bundle formula for branched coverings shows

[2] The standard reference for abelian varieties is Mumford's [443], but the short introduction [407] by Milne is also highly recommended.

[3] The same construction works in characteristic 2 under additional assumptions on A; see [283, 561]. There are fewer fixed points (4, 2, or 1), but the singularities of the quotient A/ι are worse and the minimal resolution defines a K3 surface if and only if A is not supersingular. Recently the case of $\mathrm{char}(k) = 2$ has been revisited by Schröer, Shimada, and Zhang in [532, 558].

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$\omega_X \simeq \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \simeq \mathcal{O}_X$, and, therefore, for C non-singular X is a K3 surface (of degree two), called a *double plane*.

If C is the union of six generic lines in $\mathbb{P}^2$, a local calculation reveals that the double cover X has 15 rational double points. The 15 points correspond to the pairwise intersections of the six lines. Blowing up these 15 singular points produces a K3 surface X'. The canonical bundle does not change under the blow-up; see [33, III. Prop. 3.5].

2 Classical Invariants

We start by recalling basic facts on the intersection pairing of divisors on general smooth surfaces before specializing to the case of K3 surfaces.

2.1 Let X be an arbitrary non-singular complete surface over k. For line bundles $L_1, L_2 \in \text{Pic}(X)$ the intersection form $(L_1.L_2)$ can be defined as the coefficient of $n_1 \cdot n_2$ in the polynomial $\chi(X, L_1^{n_1} \otimes L_2^{n_2})$ (Kleiman's definition; see [235, I. Sec. 5]) or, more directly, as (see [437, Lect. 12])

$$(L_1.L_2) \coloneqq \chi(X, \mathcal{O}_X) - \chi(X, L_1^*) - \chi(X, L_2^*) + \chi(X, L_1^* \otimes L_2^*). \tag{2.1}$$

Of course, both definitions define the same symmetric bilinear form with the following properties:

(i) If $L_1 = \mathcal{O}(C)$ for some (e.g. for simplicity integral) curve $C \subset X$, then $(L_1.L_2) = \deg(L_2|_C)$.

(ii) If $L_i = \mathcal{O}(C_i)$ for two curves $C_i \subset X$, $i = 1, 2$, intersecting in only finitely many points $x_1, \ldots, x_n$, then

$$(L_1.L_2) = \sum_{i=1}^{n} \dim_k(\mathcal{O}_{X,x_i}/(f_{1,x_i}, f_{2,x_i})).$$

Here, f_{1,x_i}, f_{2,x_i} are the local equations for C_1 and C_2, respectively, in x_i.

(iii) If L_1 is ample and $L_2 = \mathcal{O}(C)$ for a curve $C \subset X$, then

$$(L_1.L_2) = (L_1.C) = \deg(L_1|_C) > 0. \tag{2.2}$$

(iv) The Riemann–Roch theorem for line bundles on surfaces asserts:[4]

$$\chi(X, L) = \frac{(L.L \otimes \omega_X^*)}{2} + \chi(X, \mathcal{O}_X). \tag{2.3}$$

[4] Of course, this is a special case of the much more general Hirzebruch–Riemann–Roch theorem (or of the even more general Grothendieck–Riemann–Roch theorem), but a direct, much easier proof exists in the present situation; see [437, Lect. 12] or [236, V.1].

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We often write $(L.C)$ and $(C_1.C_2)$ instead of $(L.\mathcal{O}(C))$ and $(\mathcal{O}(C_1).\mathcal{O}(C_2))$ for curves or divisors C, C_i on X. Instead of $(L.L)$, we often use $(L)^2$ and similarly $(C)^2$ instead of $(C.C)$.

The *Néron–Severi group* of an algebraic surface X is the quotient

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$$

by the connected component of the Picard variety $\mathrm{Pic}(X)$, i.e. by the subgroup of line bundles that are algebraically equivalent to zero.

A line bundle L is *numerically trivial* if $(L.L') = 0$ for all line bundles L'. For example, any $L \in \mathrm{Pic}^0(X)$ is numerically trivial. The subgroup of all numerically trivial line bundles is denoted $\mathrm{Pic}(X)^\tau \subset \mathrm{Pic}(X)$ and yields a quotient of $\mathrm{NS}(X)$

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X).$$

Clearly, $\mathrm{Num}(X)$ is a free abelian group endowed with a non-degenerate, symmetric pairing:

$$(\ .\ ) : \mathrm{Num}(X) \times \mathrm{Num}(X) \longrightarrow \mathbb{Z}.$$

Proposition 2.1 *The Néron–Severi group* $\mathrm{NS}(X)$ *and its quotient* $\mathrm{Num}(X)$ *are finitely generated. The rank of* $\mathrm{NS}(X)$ *is called the* Picard number $\rho(X) = \mathrm{rk}\,\mathrm{NS}(X)$.[5]

2.2 The signature of the intersection form on $\mathrm{Num}(X)$ is $(1, \rho(X) - 1)$. This is called the *Hodge index theorem*; cf. e.g. [236, V.Thm. 1.9]. Thus, $(\ .\ )$ on

$$\mathrm{NS}(X)_\mathbb{R} := \mathrm{NS}(X) \otimes_\mathbb{Z} \mathbb{R}$$

can be diagonalized with entries $(1, -1, \ldots, -1)$.

Remark 2.2 The Hodge index theorem has the following immediate consequences.

(i) The cone of all classes $L \in \mathrm{NS}(X)_\mathbb{R}$ with $(L)^2 > 0$ has two connected components. The *positive cone* $\mathcal{C}_X \subset \mathrm{NS}(X)_\mathbb{R}$ is defined as the connected component that is distinguished by the property that it contains an ample line bundle. See Chapter 8 for more on the positive cone of K3 surfaces.

(ii) If L_1 and L_2 are line bundles such that $(L_1)^2 \geq 0$, then

$$(L_1)^2(L_2)^2 \leq (L_1.L_2)^2. \tag{2.4}$$

[5] In [348] Lang and Néron gave a simplified proof of Néron's original result. To prove that $\mathrm{Num}(X)$ is finitely generated, one can use an appropriate cohomology theory. See Section 3.2 for an argument in the complex setting. Numerically trivial line bundles form a bounded family, and, therefore, $\mathrm{NS}(X) \longrightarrow \mathrm{Num}(X)$ has finite kernel and, in particular, $\mathrm{NS}(X)$ is finitely generated as well. Also, $\rho(X) = \mathrm{rk}\,\mathrm{NS}(X) = \mathrm{rk}\,\mathrm{Num}(X)$.

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Just apply the Hodge index theorem to the linear combination $(L_1)^2L_2 - (L_1.L_2)L_1$ (written additively) which is orthogonal to L_1. Note that (2.4) is simply expressing the fact that the determinant of the intersection matrix

$$\begin{pmatrix} (L_1)^2 & (L_1.L_2) \\ (L_1.L_2) & (L_2)^2 \end{pmatrix}$$

is non-positive.

2.3 For a K3 surface X one has by definition $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. Moreover, by Serre duality $H^2(X, \mathcal{O}_X) \simeq H^0(X, \omega_X)^*$ and hence $h^2(X, \mathcal{O}_X) = 1$.[6] Therefore,

$$\chi(X, \mathcal{O}_X) = 2.$$

Remark 2.3 This can be used to prove that the (algebraic) fundamental group $\pi_1(X)$ of a K3 surface X over a separably closed field k is trivial. Indeed, if $\widetilde{X} \twoheadrightarrow X$ is an irreducible étale cover of finite degree d, then $\widetilde{X}$ is a smooth complete surface over k with trivial canonical bundle such that

$$\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = d\,\chi(X, \mathcal{O}_X) = 2\,d$$

and $h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = h^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 1$ (use Serre duality). Combined this yields $2 - h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 2\,d$ and hence $d = 1$.

The Riemann–Roch formula (2.3) for a line bundle L on a K3 surface X reads

$$\chi(X, L) = \frac{(L)^2}{2} + 2. \tag{2.5}$$

Recall that a line bundle L is trivial if and only if $H^0(X, L)$ and $H^0(X, L^*)$ are both non-trivial. Thus, as Serre duality for a line bundle L shows $H^2(X, L) \simeq H^0(X, L^*)^*$, the Riemann–Roch formula for non-trivial line bundles L expresses $h^0(X, L) - h^1(X, L)$ or $h^0(X, L^*) - h^1(X, L)$.

Also note that for an ample line bundle L the first cohomology $H^1(X, L)$ vanishes (we comment on this in Theorem **2**.1.8 and Remark **2**.1.9) and hence (2.5) computes directly the number of global sections of an ample line bundle L:

$$h^0(X, L) = \frac{(L)^2}{2} + 2.$$

[6] In [236] Serre duality is proved over algebraically closed fields, but it holds true more generally. The pairing is compatible with base change, so one can pass to algebraically closed fields once the trace map is shown to exist over k. In fact, the trace map exists in much broader generality; see Hartshorne [234]. For our purposes working with Serre duality over an algebraically closed field is enough: by flat base change $H^2(X, \mathcal{O}_X) \otimes \bar{k} = H^2(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}})$ and $X_{\bar{k}}$ is again a K3 surface.

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Proposition 2.4 *For a K3 surface X the natural surjections are isomorphisms*[7]

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Num}(X).$$

Moreover, the intersection pairing $(.)$ *on* $\mathrm{Pic}(X)$ *is even, non-degenerate, and of signature* $(1, \rho(X) - 1)$.

Proof Suppose L is non-trivial, but $(L.L') = 0$ for an ample line bundle L'. Then $H^0(X, L) = 0$ and $H^2(X, L) \simeq H^0(X, L^*)^* = 0$ by (2.2). Therefore, (2.5) yields $0 \geq \chi(X, L) = (1/2)(L)^2 + 2$ and thus $(L)^2 < 0$. In particular, L cannot be numerically trivial and, hence, $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Num}(X)$. Moreover, the intersection form is negative definite on the orthogonal complement of any ample line bundle, which proves the claim on the signature. Finally, the Riemann–Roch formula $(L)^2 = 2\,\chi(X, L) - 4 \equiv 0\,(2)$ shows that the pairing is even. □

For a K3 surface X the lattice $(\mathrm{NS}(X), (\,.\,))$ is thus even and non-degenerate, but rarely unimodular. For more information about lattices that can be realized as Néron–Severi lattices of K3 surfaces, see Section **14**.3.1 and Chapter 17.

Remark 2.5 Even without using the existence of an ample line bundle, one can show that there are no non-trivial torsion line bundles on K3 surfaces. Indeed, if L is torsion, then by the Riemann–Roch formula $\chi(L) = 2$ and hence L (or its dual) is effective. However, if $0 \neq s \in H^0(X, L)$, then $0 \neq s^k \in H^0(X, L^k)$ for all $k > 0$, and, moreover, the zero sets of both sections coincide. Thus, if L^k is trivial, L is also trivial. The argument also applies to (non-projective) complex K3 surfaces.

The non-existence of torsion line bundle can also be related to the triviality of the (algebraic) fundamental group $\pi_1(X)$; see Remark 2.3. Indeed, the usual unbranched covering construction (see e.g. [33, I.17]) would define for any line bundle L of order d (not divisible by $\mathrm{char}(k)$) a non-singular étale covering $\widetilde{X} \longrightarrow X$.

2.4 We shall next explain how to use the general Hirzebruch–Riemann–Roch formula to determine the Chern number $\mathrm{c}_2(X)$ and the Hodge numbers

$$h^{p,q}(X) := \dim H^q(X, \Omega_X^p)$$

of a K3 surface X.

For a locally free sheaf (or an arbitrary coherent) sheaf F on a K3 surface X the Hirzebruch–Riemann–Roch formula reads

$$\chi(X, F) = \int \mathrm{ch}(F)\,\mathrm{td}(X) = \mathrm{ch}_2(F) + 2\,\mathrm{rk}(F). \tag{2.6}$$

[7] *Warning*: The second isomorphism does not hold for general complex K3 surfaces; see Section 3.2 and Example **3**.3.2.

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The general version of this formula can be found e.g. in [236, App. A]. For $F = \mathcal{O}_X$ the first equality is the *Noether formula*

$$\chi(X, \mathcal{O}_X) = \frac{c_1^2(X) + c_2(X)}{12} = \frac{c_2(X)}{12}$$

which yields $c_2(X) = 24$.

Next, by definition one knows $h^{p,q}(X) = 1$ for $(p, q) = (0, 0), (0, 2), (2, 0), (2, 2)$ and $h^{0,1}(X) = 0$ for any K3 surface. For the remaining Hodge numbers, (2.6) implies

$$2\, h^0(X, \Omega_X) - h^1(X, \Omega_X) = \mathrm{ch}_2(\Omega_X) + 4 = 4 - c_2(\Omega_X) = -20.$$

It is also known that $h^0(X, \Omega_X) = 0$ and hence $h^1(X, \Omega_X) = 20$. Using $\mathcal{T}_X \simeq \Omega_X$, this vanishing can be rephrased, maybe more geometrically, as $H^0(X, \mathcal{T}_X) = 0$, i.e. a K3 surface has no global vector fields. In positive characteristic this is a difficult theorem on which we comment later; see Sections **9**.4.1 and **9**.5.1.[8] For $\mathrm{char}(k) = 0$ it follows from the complex case to be discussed below and the Lefschetz principle. In any event, the *Hodge diamond* of any K3 surface looks like this:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{2.7}$$

This holds for K3 surfaces over arbitrary fields and also for non-projective complex ones; see below.

3 Complex K3 Surfaces

Even if interested solely in algebraic K3 surfaces (and maybe even only in those defined over fields of positive characteristic), one needs to study non-projective complex K3 surfaces as well. For example, the twistor space construction, used in the proof of the global Torelli theorem (see Chapter 4), which is one of the fundamental results in K3 surface theory, always involves non-projective K3 surfaces. For this reason, we try to deal simultaneously with the algebraic and the non-algebraic theory throughout this book.

[8] Note, however, that it can often easily be checked in concrete situations. For example, it is easy to see that $H^0(X, \mathcal{T}_X) = 0$ for smooth quartics $X \subset \mathbb{P}^3$, complete intersection K3 surfaces, and Kummer surfaces (for the latter, see [28, Rem. 10.7]). Thanks to Christian Liedtke for pointing this out.

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3.1 The parallel theory in the realm of complex manifolds starts with the following definition.

Definition 3.1 A *complex K3 surface* is a compact connected complex manifold X of dimension two such that $\Omega_X^2 \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Serre's GAGA principle (see [546, 446]) allows one to associate with any scheme of finite type over $\mathbb{C}$ a complex space $X^{\rm an}$ whose underlying set of points is just the set of all closed points of X. Moreover, with any coherent sheaf F on X there is naturally associated a coherent sheaf $F^{\rm an}$ on $X^{\rm an}$. These constructions are well behaved in the sense that, for example, $\mathcal{O}_X^{\rm an} \simeq \mathcal{O}_{X^{\rm an}}$ and $\Omega_{X/\mathbb{C}}^{\rm an} \simeq \Omega_{X^{\rm an}}$. Also, there exists a natural morphism of ringed spaces $X^{\rm an} \longrightarrow X$.

For X projective (proper is enough) the construction leads to an equivalence of abelian categories

$$\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X^{\rm an}).$$

In particular, $H^*(X, F) \simeq H^*(X^{\rm an}, F^{\rm an})$ for all coherent sheaves F on X and smoothness of X implies that $X^{\rm an}$ is a manifold.

These general facts immediately yield the following proposition:

Proposition 3.2 *If X is an algebraic K3 surface over $k = \mathbb{C}$, then the associated complex space $X^{\rm an}$ is a complex K3 surface.*

It is important to note that all complex K3 surfaces obtained in this way are projective, but that there are (many) complex K3 surfaces that are not. In this sense we obtain a proper full embedding

$$\{\text{ algebraic K3 surfaces over } \mathbb{C}\,\} \hookrightarrow \{\text{ complex K3 surfaces }\}.$$

The image consists of all complex K3 surfaces that are projective, i.e. that can be embedded into a projective space. This is again a consequence of GAGA, because the ideal sheaf of $X \subset \mathbb{P}^n$ is a coherent analytic sheaf and hence associated with an algebraic ideal sheaf defining an algebraic K3 surface. A natural question at this point is whether complex K3 surfaces are at least always Kähler. This is in fact true and of great importance, but not easy to prove. See Section **7**.3.2.

Example 3.3 The constructions described in the algebraic setting in Example 1.3 work as well here. They define different incarnations of the same geometric objects. Only for Kummer surfaces do we gain some flexibility by working with complex manifolds. Indeed, abelian surfaces A (over $\mathbb{C}$) can be replaced by arbitrary complex tori of dimension two, i.e. complex manifolds of the form $A = \mathbb{C}^2/\Gamma$ with $\Gamma \subset \mathbb{C}^2$ a lattice of rank four. The surface X, obtained as the minimal resolution of A/ι or, equivalently, as the quotient of the blow-up of all two-torsion points $\widetilde{A} \longrightarrow A$ by the lift

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$\tilde{\iota}$ of the canonical involution, is a complex K3 surface. Indeed all algebraic arguments explained in Example 1.3, (iii), work in the complex setting.

One can show that X is projective if and only if the torus A is projective, i.e. the complex manifold associated with an abelian surface. It is known that many (in some sense most) complex tori $\mathbb{C}^2/\Gamma$ are not projective; cf. [64, 139]. Thus, we obtain many K3 surfaces this way that really are not projective.

Describing other examples of non-projective K3 surfaces is very difficult, which reflects a general construction problem in complex geometry.

3.2 Many but not all of the remarks and computations in Section 2 are valid for arbitrary complex K3 surfaces. For complex K3 surfaces, however, we have in addition at our disposal singular cohomology which sheds a new light on some of the results.

First, the long cohomology sequence of the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^*) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow$$
$$\longrightarrow H^2(X,\mathcal{O}) \longrightarrow H^2(X,\mathcal{O}^*) \longrightarrow H^3(X,\mathbb{Z}) \longrightarrow 0$$

which for a complex K3 surface X (where $H^1(X,\mathcal{O}) = 0$) shows

$$H^1(X,\mathbb{Z}) = 0$$

and by Poincaré duality also $H^3(X,\mathbb{Z}) = 0$ up to torsion. So, in addition to $H^0(X,\mathbb{Z}) \simeq H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$, the only other non-trivial integral singular cohomology group of X is $H^2(X,\mathbb{Z})$. We come back to the computation of its rank presently.

From the above sequence and the usual isomorphism $\mathrm{Pic}(X) \simeq H^1(X,\mathcal{O}^*)$, one also obtains the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}). \tag{3.1}$$

In other words, $\mathrm{Pic}(X)$ can be realized as the kernel of $H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O})$. As $\mathbb{C} \simeq H^2(X,\mathcal{O})$ and by Remark 2.5 also $\mathrm{Pic}(X)$ are both torsion free, one finds that also $H^2(X,\mathbb{Z})$ is torsion free. A standard fact in topology says that the torsion of $H^i(X,\mathbb{Z})$ can be identified with the torsion of $H^{\dim_{\mathbb{R}} X - i + 1}(X,\mathbb{Z})$, which in our case shows that $H^3(X,\mathbb{Z})$ is indeed trivial (and not only up to torsion).

The intersection form $(.)$ on $\mathrm{Pic}(X)$ is defined as in the algebraic case. In the complex setting it corresponds, under the above embedding $\mathrm{Pic}(X) \hookrightarrow H^2(X,\mathbb{Z})$, to the topological intersection form on $H^2(X,\mathbb{Z})$. The inclusion also shows that

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{NS}(X)$$

holds for complex K3 surfaces as well; cf. Proposition 2.4.

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