1

Financial Mathematics Principles

1.1 Financial Derivatives and Derivatives Markets

A financial derivative is a special type of financial contract whose value and payouts depend on the performance of a more fundamental underlying asset. One finds derivatives on the basis of all kinds of underlying entities, such as equity derivatives, whose performance is linked to the behaviour of underlying stock prices or stock indices; fixed income derivatives, whose payout depends on the level of interest rates; currency derivatives, that are connected with one or more FX exchange rates; derivatives on commodities which, for example, can depend on the (joint) evolution of oil, gas, gold, orange juice or any other commodity prices. Actually, they can come in all forms and on the basis of all kinds of underliers; sometimes even a combination of several underliers from different asset classes (hybrids).

Derivatives come into existence in modern economies and encourage price discovery in free markets with consequent price volatility. Often, good business planning requires some limited price stability. This can be (partially) provided by derivatives. One then shifts price risk to professionals better positioned to manage the oscillations. Derivatives can be used for many purposes. They can be used not only to mitigate price risk but also to speculate on it. Many contracts and structured products are implemented using derivatives. A sophisticated risk-management of a portfolio uses derivatives to hedge away as completely as possible all undesired exposure. Some capital instruments even have derivative features and can be seen essentially as cash flows being made contingent on the resolution of future uncertainties.

Derivatives are traded on exchanges (like the CBOE) or over the counter (OTC). Derivatives traded on exchanges are typically standardized; OTC derivatives are tailor made. You can compare it with a clothes collection that comes in one design and is available in standardized sizes (XS, S, M, L, XL, . . .) and colours. These you find in retail shops (compare with exchanges). If, however, you desire a different design, which fits your frame and specific body form and is printed in your preferred (non-standard) colour, you have to go to a tailor and have it custom made (compare with OTC).

Both forms have their advantages and disadvantages. You buy the standardized contract via the exchange and you neither know nor care about the actual counterparty who is selling the derivative to you (this is like knowing the shop but not the person/tailor who actually made your clothes). It is the exchange that takes care that the obligations in the contract are met and the exchange is taking the risk of counterparty default while simultaneously putting in place measures against such default. As long as the exchange itself doesn't default, the contract is honoured, as far as you are concerned. This is different with OTC derivatives; here there is no intermediator and you are dealing directly with your counterparty (this is like interacting with the tailor who is actually making your clothes). A default of this counterparty can lead to huge losses since, no matter what was agreed, a default of your counterpart can mean non-delivery of what has been promised. Of course, OTC derivatives can be designed to your very specific needs and may better suit you than a standardized exchange-traded product. On the other side, exchange-traded products are typically more liquid and can be bought and sold back easily; there are continuously bid and ask prices quoted at which you can immediately transact. The bid price is the price at which you can sell: the price that somebody is bidding for the asset. The ask price is the price at which you can buy: the price that somebody is asking for it. The more common the product is, the lower the spread, i.e. the difference between the ask and the bid price. It is much harder to unwind or sell back an OTC derivative. It can actually involve a new negotiation round with unclear terms of settlement. Maybe your counterparty is not even willing to take it back, and even in case that it is, the spread is usually much higher than the spread of exchange-traded products because the product is very specific and exotic. It involves many more uncertainties (model risk, calibration risk, ...), its hedging is more complex (and hence it is also more involved for your counterparty to unwind it) and more (safety) margin is charged. It is just a deal between a very limited number of parties, and therefore the ask and demand forces are not in place as they would be on exchanges open to everybody on the globe.

Derivatives are omnipresent in today's financial markets. There are various types. One has futures and forwards, which are basically contracts to buy or sell the underlying at a predetermined price in the future on a predetermined date. Hence the limited price stability offered. One also has swaps, that agree on exchanging certain uncertain cash flows over a predetermined period, and finally one has options, that are agreements in which the holder has a right to buy or sell the underlying at specific conditions. To buy an option, one has to pay the price or the premium for the option; swaps are often initiated at zero cost and have therefore an initial market price equal to zero. However, they can be written on a huge underlying notional and, although of zero value when the deal is struck, can bear significant risks. The actual size of derivatives' markets is not easy to estimate, but at the end of 2013, the Bank for International Settlements (BIS) estimated the total notional outstanding for OTC derivatives at USD 710 trillion and at a USD 18.6 trillion gross market value.

The theory we develop in this book, on how to determine the bid and ask prices and how to deal with the risks of such derivatives, is applicable to all derivative types over all asset classes mentioned. However, we mainly focus on equity derivative options. The basic examples are European Call and Put options, often referred to as vanilla options because they can be regarded as the most simple type of options. Before we define them and illustrate their use in Section 1.3, we first recall the concept of the risk-free bank account.

1.2 The Risk-Free Bank Account and Discount Factors

A risk-free interest rate can be viewed as the interest rate that rewards the depositor of some amount of money on lending it to a counterparty that cannot default. Traditionally, "risk-free" interest rates were derived from the rates associated with Treasury bonds issued by governments. However, nothing is actually without any risk of default, and history has shown that some governments can and have defaulted on their obligations (Argentina, Greece, Cyprus, . . .).

If you deposit USD 1000 today in a risk-free account earning an interest rate of $r = 2\%$ (per annum), then the value in one year's time would be USD 1020 ($= 1000(1 + 0.02)$). In this calculation we have assumed that the interest is compounded annually.

Now assume that you want to have exactly USD 2000 (N) in the account in three years' time (T) from now, and assume we have a (flat) interest rate of $r = 2\%$ (per annum). How much do you have to put into the risk-free account now? The answer is given by the formula:

$$\frac{N}{(1+r)^T} = \frac{2000}{(1+0.02)^3} = 1884.64.$$

Indeed, after putting USD 1884.64 into the account, it grows under a 2% interest rate after one year to USD 1922.33 ($= 1884.64(1+0.02)$) and after another year to USD 1960.78 ($= 1922.33(1 + 0.02)$) and finally, after the last year, to USD 2000 ($= 1960.78(1 + 0.02)$). The interest is compounded again annually, and we call USD 1884.64 the present value of USD 2000 received in three years. The ratio of both is called the discount factor or the price of future money and is equal to

$$\frac{1}{(1+r)^T}.$$

In our example it equals 0.9423. It is basically today's value of receiving USD 1 in T years from now.

In the above examples, we had annual compounding; i.e. interest is paid after each year. One could also have other schemes of interest payment, such as semi-annually, quarterly, monthly, weekly and daily. In general, then, N units of currency will grow to

$$N\left(1+\frac{r}{m}\right)^{mT}$$

if we compound m times per year and keep the money in our account until time T. The discount factor then equals

$$\left(1+\frac{r}{m}\right)^{-mT}.$$

For example, if we have quarterly compounding, $m = 4$ and $r = 2\%$, and we again put 1000 on the account for one year, the account grows after a first interest payment to 1005 ($= 1000(1 + 0.005)$) after three months. After three other interest payments, one at six months, one at nine months and one after a year, it would accumulate at the end of year one to 1020.15 ($= 1000(1+0.02/4)^4$), which is just slightly more than with annual compounding. The obvious reason is that, after an interest payment is made, the investor starts to earn interest on this payment. Note that different compounding conventions lead to different rate quotes consistent with the same price for future money at a fixed future date.

When pricing derivatives, we usually assume continuous compounding of interest rates, meaning that compounding (i.e. receiving interest) occurs continuously, i.e. over an infinitesimally small period of time, or in other words if $m \to \infty$. Our discount factor then becomes

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{-mT} = \exp(-rT).$$

In Figure 1.1, one clearly sees the convergence if $m \to \infty$; one can also see the differences between yearly ($m = 1$), semi-annually ($m = 2$), quarterly ($m = 4$), monthly ($m = 12$) and weekly ($m = 52$) compounding on an investment of USD 1000 at $r = 2\%$ during a period of exactly one year ($T = 1$).

In reality, each maturity T and quoting convention has its own interest rate, $r(T)$ say, reflecting the market expectation of changing interest rates over the given period. We then speak about a yield curve and the interest rate term structure.

The risk-free yield curve is a curve showing several yields or risk-free interest rates across different contract lengths (maturities), known as the "term", for a risk-free debt contract. One has different curves for different currencies. For example,

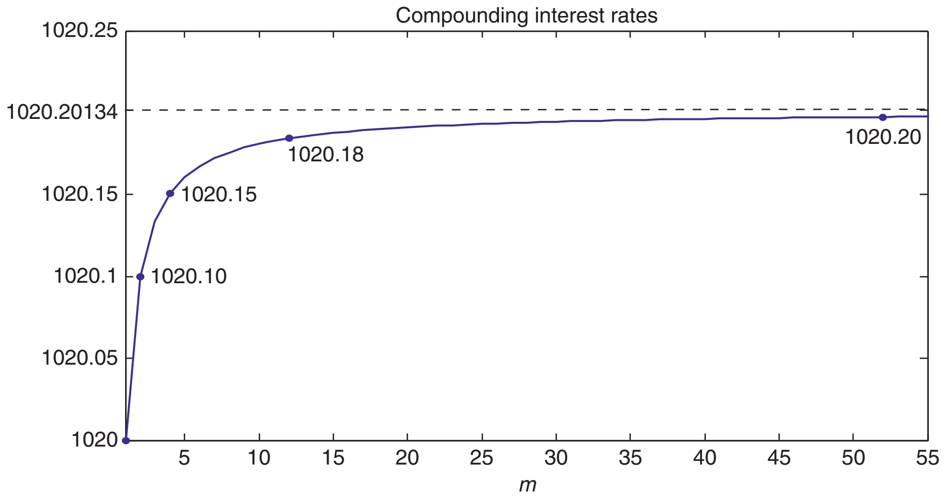

Figure 1.1 Compounding interest rates ($T = 1, N = 1000, r = 2\%, m = 1, 2, 4, 12, 52$)

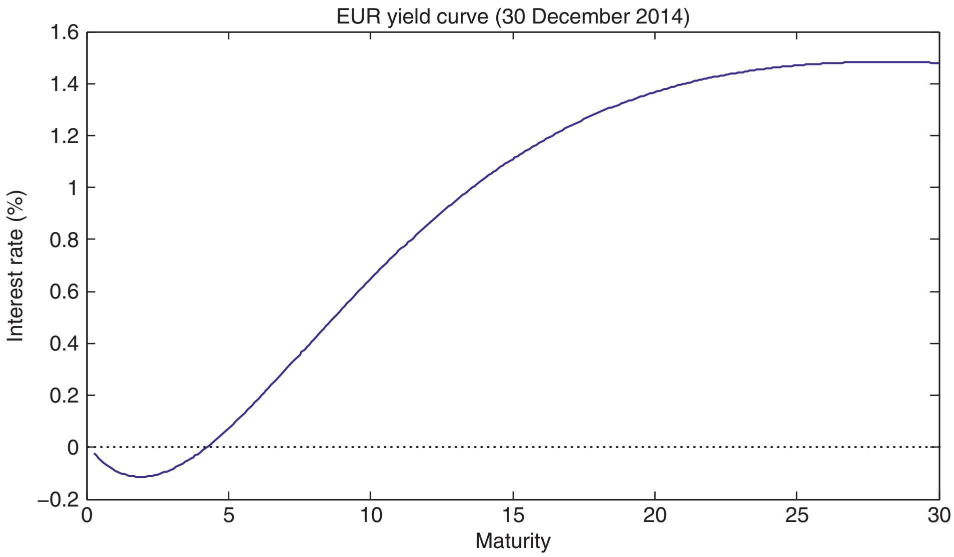

Figure 1.2 EUR yield curve (30 December 2014)

one has the US dollar yield curve based on interest rates paid on US Treasury securities for various maturities, which are assumed to be (almost or as close as possible to) risk-free. In Figure 1.2 the EUR yield curve is shown as of 30 December 2014; this curve is calculated by the ECB and based on "AAA-rated" Euro-area central government bonds.

Interest rate curves are typically upward-sloping, with shorter-term interest rates lower than longer-term interest rates, but can also be downward-sloping (inverted) or humped. An inverted curve, for example, would indicate that the market expects lower interest rates in the future. As seen in the EUR-curve of Figure 1.2, yields can be negative, meaning investors are basically paying money to park their investments in the underlying securities.

Related to a yield curve is a discount curve. The discount curve basically represents the discount factors for the different terms. For a given term, the discount factor is the present value of a currency unit promised at the given term. If $r(t)$ is the yield (continuously compound) associated with the maturity (or term) t, then the discount factor for that term equals $D(t) = \exp(-r(t)t)$. Figure 1.3 shows the discount factors based on the yields of Figure 1.2. From this one can see that, on 30 December 2014, receiving EUR 100 in 30 years' time from that point would be equivalent to receiving EUR 64.16 on that day, since the 30-year interest rate on 30 December 2014 was $r(30) = 0.0147947$. Similarly, since the one-year interest rate was then $r(1) = -0.0008869$, investors need to pay about EUR 100.09 on 30 December 2014 to receive EUR 100 back on 30 December 2015.

Discount factors are used to discount cash flows (at the risk-free rate). Assume, for example, a cash flow consisting of N payments of EUR C_i paid out at times t_i, $i = 1, \ldots, N$. Then the present value (PV) of this cash flow equals:

$$PV = \sum_{i=1}^{N} C_i \exp(-r(t_i)t_i) = \sum_{i=1}^{N} C_i D(t_i),$$

with $D(t)$ denoting here the EUR-related discount factor with term t.

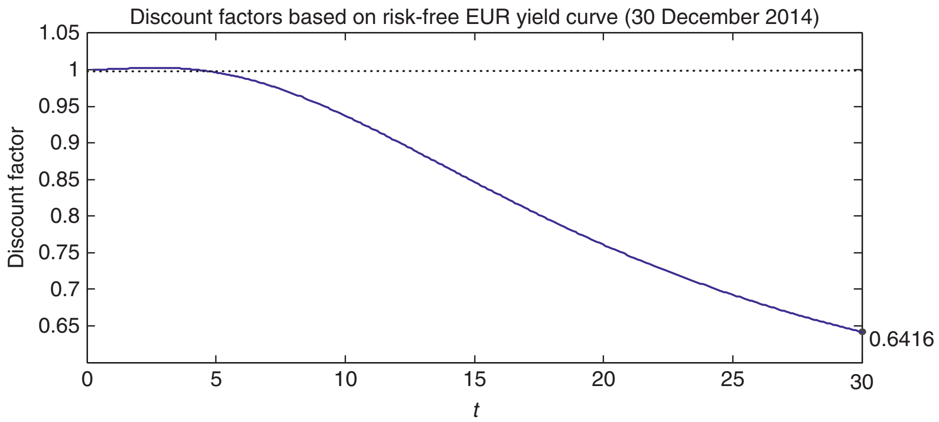

Figure 1.3 Discount factors (EUR, 30 December 2014)

1.3 Vanilla Options

A *European Call option* gives the right to the holder (the buyer) of the option to buy from the writer of the option (the seller) at a predetermined future time point $t = T$, called the maturity, the underlying, a stock, S say, for a predetermined price K, called the strike. A *European Put option* is similar, but it gives the right to the holder of the option to sell the underlying to the writer of the option at a predetermined future time point $t = T$ (maturity) for a predetermined price K (strike). For such a right the buyer pays a premium to the option seller, when the deal is initiated, say at $t = 0$. We refer to the instrument as well as its premium by the notation $EC(K, T)$ and $EP(K, T)$ respectively. It should be clear from the context whether we are referring to the instrument or the premium.

Besides these so-called *European* options, *American* Call and Put options also exist. These are different from their European counterparts in the sense that the holder can exercise his rights not only at the maturity of the option but during the entire lifetime of the option.

If an option is traded on an exchange, typically bid and ask prices are given continuously. We use the notation $bidEC(K, T)$ and $askEC(K, T)$ for the European Call's bid and ask prices and correspondingly $bidEP(K, T)$ and $askEP(K, T)$ for the European Put. It should be clear from the context whether we are referring to the initial time $t = 0$ price or to a general time t price, with $0 \leq t \leq T$.

The holder of a call option will exercise his right to buy the underlying via the option contract only if this is beneficial to him. This happens when the underlying at maturity ($t = T$) has a market price $S(T)$ that is greater than the strike price: $S(T) > K$. The payoff of the option is then strictly positive and equals the difference $S(T) - K$. In the other situation, it would not be rational to pay via the option the strike price K, which is more than the price $S(T)$ one pays in the market, and the option contract expires, worthless. The payoff is then zero. Summarizing, the payoff of the call option can in general be given by

$$\text{payoff of } EC(K, T): \ \max(S(T) - K, 0) = (S(T) - K)^+.$$

Similar reasoning can be given for the put option, where the holder will actually only sell the underlying via the option contract when at maturity ($t = T$) its actual market price $S(T)$ is lower than the strike price K: $S(T) < K$. The payoff of the put equals in general

$$\text{payoff of } EP(K, T): \max(K - S(T), 0) = (K - S(T))^+.$$

In Figures 1.4 and 1.5 the payoff functions of a European Call and a European Put, respectively, are visualized.

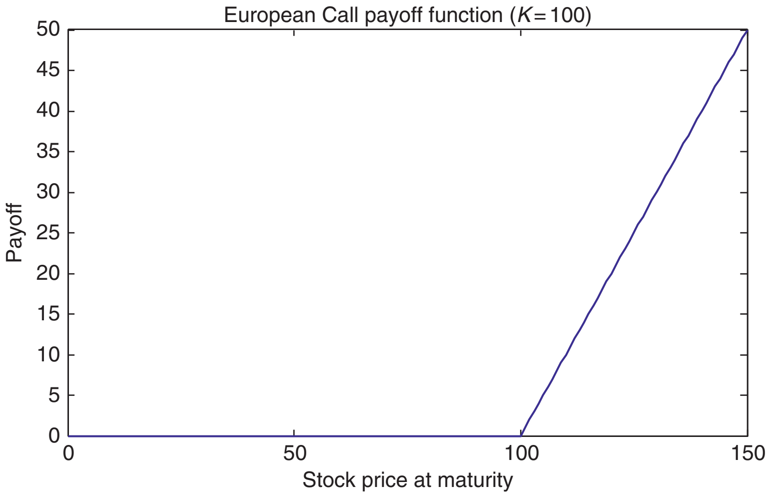

Figure 1.4 European Call (EC) payoff function ($K = 100$)

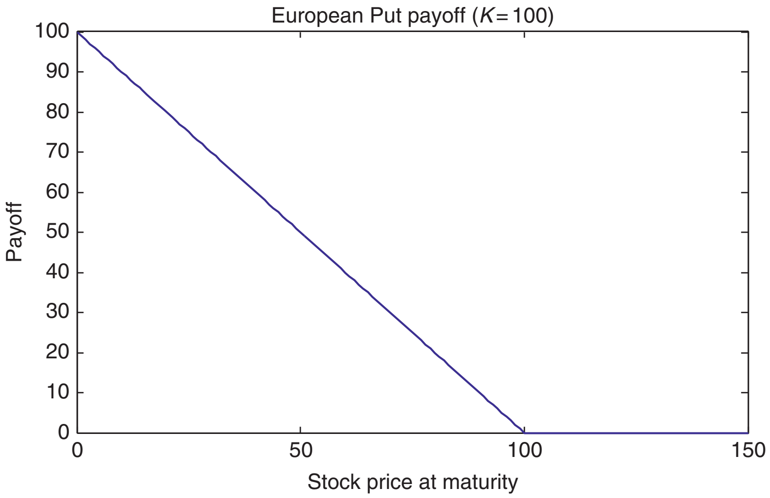

Figure 1.5 European Put (EP) payoff function ($K = 100$)

We say that a European Call option is *out of the money (OTM)* if the current stock price is below the strike. If it is above the strike we say it is *in the money (ITM)*. When the underlying stock prices equals (or is very close to) the strike we say it is *at the money (ATM)*. For a European Put, *out of the money (OTM)* corresponds to the situation where the current stock price is above the strike. If it is below the strike we say it is *in the money (ITM)*. When its stock price equals (or is very close to) the strike we again say it is *at the money (ATM)*. OTM basically means that if one were now to exercise the option (which is actually not allowed in reality for European options, but one assumes one does), the payoff would be zero. ITM means we would then have a non-zero payoff. The amount that one would get if the option were now to be exercised is called the *intrinsic value* of the option. OTM options have a zero intrinsic value. The difference between the current price of the option and its intrinsic value is often referred to as the *time value*: the extra value the option carries because there is still time left until expiry/maturity and the underlying asset can still move in a beneficial direction for the option holder. OTM options have only time value.

Put and call options can be used for different purposes. We illustrate this in the next examples. In the first example, we show how a call option can be used for speculation. The next example shows how a put can be used as a kind of insurance against downward market movements. The third example shows how a call can be used as building block in a very common structured product, a Principal Protected Note (PPN).

Example 1.1 *An investor has USD 10,000 at his disposal and is clearly convinced the stock S will rally in the next year. The stock trades now at USD 50. Vanilla options on the stock are trading as well. The one-year at-the-money (ATM), i.e. with strike equal to the current stock price, European Call option has a bid price of USD 3.85 and an ask price of USD 4.00. With his USD 10,000, the investor can hence buy 200 stocks, or he can buy 2500 European ATM call options (or he could buy some of both). He decides to buy 2500 ATM calls. Note that he is buying and hence has to pay the ask price of USD 4.00. After six months the stock has indeed rallied to USD 65. The call options he bought are now deep in the money, i.e. the current stock price is higher than the strike. They are also closer to maturity (six months) and one call trades at a bid equal to USD 18.00 and has an ask price of USD 18.25. He decides to close his position and sells his 2500 calls at the bid price of USD 18.00. He cashes USD 45,000 and actually makes a 350% profit. Compared with a direct investment in the stock this is much better, since that would have given him only a 30% return.*

Derivatives can be used for speculation and can lead to hugely leveraged positions and hence huge gains. Of course, there is another side to the story. If the stock had not moved higher and, for example, closed after one year at the same level of USD 50, the investment in call options would have led to a 100% loss, since all the call options would have been worthless at maturity. The direct investment in the stock would, in that case, have ended flat and would have shown no loss.

Example 1.2 *At the beginning of a new year an investor steps into the equity market and buys 1000 stocks at a price of USD 65 each. He furthermore has a cash account of USD 2000. His initial total wealth is USD 67,000. After nine months,* *the stock trades at USD 80. He doesn't want to exit his position because he still believes there is upwards potential in the stock. However, he is also worried about the downside. During the first nine months of the year the stock has been rallying nicely (at the beginning of the year he entered at USD 65!) and it is now vulnerable to potential negative market sentiment in the next months until year-end. He wants to close his year positively and seeks protection against adverse downside market movement without exiting his position and potentially missing a continuation of a rally in the stock. Derivatives are trading on the underlying stock. An out-of-the-money (OTM), i.e. with strike below the current spot, European Put with maturity of three months with strike USD 70 is trading with a bid of USD 1.80 and an ask of USD 2.00. He decides to spend his USD 2000 cash to buy protection and buys (at the ask price) 1000 three-month European Puts with strike USD 70. By year end the market indeed went down and the stock is now trading at USD 60. Without any protection (and assuming no interest payments on his cash account), he would have been down for the year by 7.46% (USD 5000/USD 67,000). However, due to his put options, he receives an additional payoff. At expiration the put options gave him a payoff of USD 10 each. He thus received USD 10,000. Compared with the value of his position in the beginning of the year (USD 65,000+USD 2000), he now has stock worth USD 60,000 and USD 10,000 in cash, or USD 70,000 in total. He is therefore up 4.48% (USD 3000/USD 67,000) for the year.*

Example 1.3 *Salespeople have learnt that their retail customers are very interested in investing their money in the medium term in the stock market, especially in some new social media companies. However they are also worried about losing their investment. The structuring team therefore designs the following structured product: a Principal Protected Note (PPN). For each USD 1000 you invest, you receive after four years your initial investment (USD 1000) plus 60% of the positive performance of a social media stock S. If the stock S were to end after four years below its value at initiation, the investor would still get his initial investment back. Denoting with $S(0)$ the initial stock price of S and with $S(4)$ the stock price after four years, the investor hence receives:*

$$PPN = 1000 + 1000 \times 60\% \times \max\left(\frac{S(4) - S(0)}{S(0)}, 0\right). \tag{1.1}$$

The final wealth with an initial investment of USD 1000 in either the PPN or in the stock is compared in Figure 1.6.

Assume that investing now USD 0.82 into a risk-free account would give in four years' time USD 1; this corresponds approximately with an interest rate of 5%. Assume $S(0) = 20$. Note that in this setting Equation (1.1) becomes

$$PPN = 1000 + 30 \times \max(S(4) - 20, 0) = 1000 + 30 \times (S(4) - 20)^+ .$$