Singularities, Bifurcations and Catastrophes

Suitable for advanced undergraduates, postgraduates and researchers, this self-contained textbook provides an introduction to the mathematics lying at the foundations of bifurcation theory. The theory is built up gradually, beginning with the well-developed approach to singularity theory through right-equivalence. The text proceeds with contact equivalence of map-germs and finally presents the path formulation of bifurcation theory. This formulation, developed partly by the author, is more general and more flexible than the original one dating from the 1980s. A series of appendices discuss standard background material, such as calculus of several variables, existence and uniqueness theorems for ODEs, and some basic material on rings and modules. Based on the author's own teaching experience, the book contains numerous examples and illustrations. The wealth of end-of-chapter problems develop and reinforce understanding of the key ideas and techniques: solutions to a selection are provided.

James Montaldi is Reader in Mathematics at University of Manchester. He has worked both in theoretical aspects of singularity theory as well as applications to dynamical systems, and co-edited the books: *Geometric Mechanics and Symmetry: The Peyresq Lectures* (Cambridge, 2005), *Peyresq Lectures in Nonlinear Systems* (2000), and *Singularity Theory and its Applications Part 1* (1991).

Singularities, Bifurcations and Catastrophes

James Montaldi
University of Manchester

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107151642
DOI: 10.1017/9781316585085

First published 2021

Printed in the United Kingdom by TJ Books Limited, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

ISBN 978-1-107-15164-2 Hardback
ISBN 978-1-316-60621-6 Paperback

Contents

Preface

How do we create mathematical models for systems that have discontinuous jumps? This was the fundamentally new question addressed by the topologist René Thom in his book on morphogenesis [111]. Although the beginnings of the mathematics of singularities had been around for a decade or so before, that book gave an important impetus to the later development of the subject.

What are singularities? In as brief a description as is possible, they are points where the hypotheses of the inverse or implicit function theorems fail. Singularity theory is of course the study of such singularities, and *bifurcation theory* and *catastrophe theory* form part of singularity theory, and deal with how aspects of functions and solutions of systems of nonlinear equations, change in families. Over the decades, singularity theory has become a very broad subject, and of necessity this text cannot cover it all. The principal aim of this book is to introduce readers to the techniques of singularity theory with an eye on applications to bifurcation theory, although specific applications to the natural sciences or other areas are not covered in any detail.

The book is divided into four parts. The first is dedicated to catastrophe theory, which we take to mean the study of (degenerate) critical points of smooth real-valued functions and their deformations. The second part of the book is on the singularity theory of nonlinear equations, studied using contact equivalence, while the third considers its applications to bifurcation theory. We begin with catastrophe theory because, while more restricted in application, it is technically more straightforward, and contains all the main ideas of the more general theories. The final part, the appendices, contains necessary background material as well as solutions to a selection of the problems that are found at the end of each chapter.

A very brief history To quote Poston and Stewart from the start of their book [98], '*in the beginning there was Thom*'. In his fundamental book on the subject [111], René Thom described how a large number of phenomena could be understood in terms of a few models of families of functions and their critical points, which he called ***catastrophes***. This is the subject that Zeeman and others dubbed *catastrophe theory*. The work of Thom contains the essence of all the ideas contained in this text.

As part of his programme, Thom provided a classification of the *elementary catastrophes* – those of codimension at most 4. In the 1970s, Vladimir Arnold [3, 6] produced a classification up to codimension 10 and proved some important results relating this classification to other areas of mathematics and especially to Dynkin diagrams and hence to Lie algebras.

Prior to Thom's work, Hassler Whitney in the 1940s [116, 117] found what can be seen as the first results of singularity theory, where he studies the 'essential' singularities for a map of an n–dimensional manifold into $\mathbb{R}^{2n-1}$. In the 1950s, Whitney classified the local structure of mappings of the plane to itself [118]. More or less concurrently, Thom [110] proved some general results about singularities of smooth maps between manifolds of general dimensions, and more particularly when the target dimension is 2.

In the late 1960s and early 1970s, following the work of Thom, John Mather wrote a series of six fundamental and technical papers [76] which moved the fledgling subject of singularities of mappings to a different level. Since then, there have been many advances and refinements, most based on the ideas of Mather, enhanced by methods of commutative algebra and algebraic geometry.

Probably the single most important application of singularity theory is to the foundations of bifurcation theory, an approach developed principally by Martin Golubitsky, starting with two seminal papers he wrote with David Schaeffer at the end of the 1970s [48], which then developed into two monographs [49, 50], the second on symmetric bifurcations written together with Ian Stewart.

Thom's idea Many physical, biological and other systems can be modelled by equations (often differential equations) depending on a number of external parameters, and it is important to understand how the solutions (or equilibrium points) of the equations depend on these external parameters. Thom considered systems governed by a potential function V to be of particular importance, where the stable states of such a system are determined by local minima of V.

As parameters are varied, the number of local minima of V can change, and in particular, if there is more than one local minimum, then the system must somehow decide which one it occupies, possibly dependent on its history. Generally speaking, a small change in the parameter values will result in a small change in the state of the system, and the system has a continuous response to the parameters. However, at some parameter values there can be abrupt changes, where the state of the system is forced to jump from one local minimum to another, and these Thom called *catastrophes*. We describe these in more detail at the end of Chapter 2.

Thom's profound insight is that it is possible to explain and model these abrupt changes, and moreover there are only a few possible models (fold, cusp, swallowtail, umbilic, etc.), and, perhaps surprisingly, which of these models are relevant to a particular system depends more on the number of parameters and less on the number of variables modelling the system. Finding the form of these models is the principal aim of catastrophe theory, and the theory predicts that these give a description of how systems 'generically' respond to changes in parameters. 'Generic' here means that these models are likely to appear in any such system, and they are robust to small changes in the model, or, from the opposite perspective, other catastrophes are unlikely to occur, and if they do, they will not be robust.

Thom himself [111], and particularly Christopher Zeeman [120, 121], developed many applications of catastrophe theory in the 1970s. Some of these were well–founded scientifically, while others (such as using the cusp catastrophe to model when dogs bark or when prisoners riot) have more dubious foundations. These less well–founded applications provoked a certain amount of criticism, and led to catastrophe theory being branded in some quarters as pseudo-science. However, these critics often missed the point Thom was making, in particular that the possible responses of a system are independent of whether the equations represent the system precisely, or even whether one has modelled all the ingredients of the system. The language of folds and cusps is now part of everyday science, vindicating, I believe, Thom's initial vision.

The text As mentioned above, the text is divided into four parts.

The first part presents catastrophe theory, and the content is not very different from that of the book by Bröcker [12], although the presentation is different, as befits a textbook for undergraduates. In this part, we prove the fundamental theorems on finite determinacy and state the versal unfolding theorem in the context of right equivalence of functions (germs). The proof of the versal unfolding theorem is given in Part II.

I should point out that my use of the term 'catastrophe theory' is not universal. I use it for the study of critical points of functions, rather than solutions of more general equations. In a single variable the distinction is not essential: if one is interested in solving an equation $f(x) = 0$, then defining V to be any integral of f (a solution of $V'(x) = f(x)$), solving the given equation is the same as finding critical points of V. In more than one variable, the two notions are of course not equivalent (such a V exists only if, as a vector field, $f(x)$ is conservative).

Part II concerns the singularity theory of solutions of systems of (nonlinear) equations, studied using the technique of contact equivalence of map germs. Again, we discuss the relevant theorems on finite determinacy and versal unfoldings. We also discuss some of the known classification of map germs up to contact equivalence. Much of this material can be found in the literature, though not in text book form. This part also includes a discussion of the Malgrange preparation theorem (in a form due to Mather), and a proof of the versality theorems. It concludes with a short digression on left–right equivalence of maps.

Part III is more novel and describes in some detail the so-called *path approach* to bifurcation theory. This approach was first considered by Golubitsky and Schaeffer in the 1970s, but the underlying singularity theory was not available at the time and so they used their distinguished parameter approach, which does work well for 1–parameter bifurcations. In the 1980s, Damon introduced a new equivalence relation, called $\mathcal{K}_V$-equivalence, and it was pointed out by your current author [85] that this can be used to define a path approach to bifurcation theory. This was followed up in particular by Furter, Sitta and Stewart in a series of papers on symmetric bifurcations; see for example [37, 38, 39] and references therein.

Part III begins with a description of this path approach to bifurcation theory in the context of a few well-known bifurcations such as the pitchfork and hysteresis bifurcations. We then develop Damon's $\mathcal{K}_V$-equivalence for map germs relative to a variety V in the target. We illustrate the method by classifying all low–codimension bifurcation problems with up to three parameters, by classifying the paths relative to the discriminant of a versal unfolding. This part finishes with a discussion of some bifurcation problems having additional structure, or constraints, which is included to illustrate the flexibility and general methodology of this path approach.

The text concludes with a series of five appendices. The first four of these contain standard background material, such as calculus of several variables, existence and uniqueness theorems for ODEs, and some basic material on rings and modules. Appendix E contains solutions to a selection of the problems given at the ends of the chapters.

What's not included While I have provided proofs of almost all results, the Malgrange preparation theorem is not proved. This central theorem is used to prove the versal unfolding theorems, and is discussed in Chapter 16; the interested reader can find proofs of it in the literature cited in that chapter.

It is customary to present singularity theory equivalence relations as the result of group actions. This is used firstly to develop techniques for classifying singularities, and secondly to provide converse results to the finite determinacy theorems. In this text, we avoid any discussion of group actions, and leave the equivalence relations as equivalence relations. For the classifications, particularly in Part III, we use an approach based on 'constant tangent spaces'. The converse results to finite determinacy (e.g. Theorem 5.17) are stated without proof.

It might be thought natural to include a discussion of manifolds as the natural spaces on which smooth maps are defined. We simplify matters by considering only submanifolds of Euclidean space (of course, with no loss of generality thanks to Whitney's embedding theorem). The material on submanifolds, along with the implicit function theorem and its friends, is presented in the second Appendix. A proof of the inverse function theorem using the homotopy method is given in Chapter 5 as a precursor for the proof, by the homotopy method, of the finite determinacy theorem for right equivalence.

Although Part III is dedicated to the study of bifurcations, we do not study dynamical properties of bifurcations in dynamical systems. Many excellent texts cover this, for example the books of Guckenheimer and Holmes [55] and Kuznetsov [67].

Throughout the text, attention is restricted to the singularity theory of smooth real maps and their germs. However, all the finite determinacy and unfolding theorems carry through for the complex analytic setting – one simply replaces the ring $\mathcal{E}_n$ of smooth germs with the ring $\mathcal{O}_n$ of holomorphic germs in both statements and proofs. The only specifically (complex) analytic results included are a summary of some of the beautiful geometric criteria for finite determinacy. I make no apology for not giving the proofs, which would require a development of sheaf theory and Hilbert's Nullstellensatz, and would be a book of its own; the reader can find some details in the survey article of Wall [115] and in the books of Ebeling [34], of Greuel, Lossen and Shustin [53] and of Mond and Nuño Ballesteros [81].

We assume throughout that all maps are C^∞. The literature contains numerous results about the cases where the maps in question are only C^r for some $r < \infty$. An important question in this direction is topological equivalence of smooth maps. It was originally hoped that stable maps would be dense (in an appropriate topology) in the space of C^r maps ($r > 0$) between any pair of manifolds, but Mather showed that this was not the case: it depends on the dimensions of the manifolds in question (Mather's 'nice dimensions' are where the stable maps are dense; see [76, VI]). Instead, it was necessary to introduce the idea of topological stability of smooth maps, namely that nearby maps were topologically equivalent to the given one. We say nothing about topological equivalence, and the interested reader can find much information in the monograph of du Plessis and Wall [94].

One of the important properties (invariants) of a singularity is its local topology, particularly in the complex setting, including for example the relation between the codimension of the singularity and the homology of its smooth fibres. This text has no discussion of this, but the interested reader will find it a central theme in the recent book by Mond and Nuño Ballesteros [81].

Another omission is symmetry. Except for brief treatments of catastrophe theory for even functions (Chapter 9) and bifurcations of odd maps (Section 23.1), this book does not deal with the interesting issue of symmetry, which is particularly important in bifurcation theory. Discussion of symmetric systems can be found in the book of Golubitsky, Stewart and Schaeffer [50] and the more recent book of Golubitsky and Stewart [51].

There are presumably other omissions omitted from this list of omissions.

Teaching the course I have taught most of Part I of this text (on critical points of functions) as a 20-hour lecture course, to students in the third year of a UK undergraduate mathematics degree programme. As a more advanced 30-hour master's-level course, I have taught the majority of Parts I and II, but never thus far, Part III. Each chapter ends with a collection of problems, and a selection of solutions are given in the final appendix. The reader will notice that the chapters for the earlier parts have many more problems (and more solutions), corresponding to the teaching experience of the author.

Technology The text was typeset with LaTeX (of course). The diagrams were produced using two pieces of software. The line drawings (e.g. Figure 7.1) were made using the LaTeX package *pstricks* developed principally by Herbert Vöss, and part of the standard LaTeX distributions. The surfaces similar to those in Figure 7.3 were produced using the software Surfer [107] for visualizing algebraic surfaces and developed under the auspices of the Mathematisches Forschungsinstitut Oberwolfach. Finally, two pieces of software were used to help with or check calculations: Macaulay2 (which is free) and Maple (which is not).

Notation The index begins with a list of notation, with the page where each symbol is introduced. The symbol := (as in $A := x$) means 'is defined to be' (A is defined to be x). Proofs end with the symbol ✔, remarks with the symbol ❞, definitions with ✯ and examples with ✐. Those problems that end with (†) have solutions, some in greater detail than others, in Appendix E.

Thanks I thank the many people whose collaborations, discussions and seminars influenced my approach to singularity theory and its applications. I also thank students of my lecture courses for comments on the original lecture notes that helped improve this final version.

From a personal historical perspective, I would like to acknowledge in particular David Mond and Mark Roberts, with whom I learned singularity theory as a PhD student at the University of Liverpool in the early 1980s. Also influential were others, staff and visitors, at Liverpool at the time, including Bill Bruce, Jim Damon, Terry Gaffney, Peter Giblin, Chris Gibson, Ian Porteous and Terry Wall. Others from whom I have learned much include Marty Golubitsky, Andrew du Plessis, Ian Stewart, Duco van Straten and Christopher Zeeman.

There are many others whose seminars and occasional conversations have also been influential and should be mentioned, but I would be afraid of missing someone out from the long list. Thanks are also due to them.

I thank in particular Terry Gaffney for discussions on Martinet's theorems which contributed to Chapter 17.

Finally, since, as I write, no–one else has seen this text, all and any errors are indeed my own, and I cannot pretend to share the blame with anyone. I just hope there are not too many.

Glossop
August, 2020