Singular Intersection Homology

Intersection homology is a version of homology theory that extends Poincaré duality and its applications to stratified spaces, such as singular varieties. This is the first comprehensive expository book-length introduction to intersection homology from the viewpoint of singular and piecewise linear chains. Recent breakthroughs have made this approach viable by providing intersection homology and cohomology versions of all the standard tools in the homology toolbox, making the subject readily accessible to graduate students and researchers in topology as well as researchers from other fields.

This text includes both new research material and new proofs of previously known results in intersection homology, as well as treatments of many classical topics in algebraic and manifold topology. Written in a detailed but expository style, this book is suitable as an introduction to intersection homology or as a thorough reference.

Greg Friedman is Professor of Mathematics at Texas Christian University. Professor Friedman's primary research is in geometric and algebraic topology, with particular emphases on stratified spaces and high-dimensional knot theory. He has given introductory lecture series on intersection homology at the University of Lille and the Fields Institute for Research in Mathematical Sciences. He has received grants from the National Science Foundation and the Simons Foundation.

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Preface

This book arose thanks to a short course the author was asked to give in Lille in 2013 as an introduction to intersection homology theory. Originally conceived as a set of written lecture notes, the project quickly grew into the more comprehensive volume that follows. The goal has been to provide a single coherent exposition of the basic PL (piecewise linear) and singular chain intersection homology theory as it has come to exist today. Older results have been given more detailed treatments than previously existed in the literature, and several newer, though likely not unexpected, topics have been newly developed here, such as intersection homology Poincaré duality and products over Dedekind rings, including $\mathbb{Z}$.

To say a word about our primary topic, though a more extensive introduction will be provided in Chapter 1, intersection homology was first developed by Mark Goresky and Robert MacPherson in the late 1970s and early 1980s in order to generalize to spaces with singularities some of the most significant tools of manifold theory, including Poincaré duality and signatures. Although originally introduced in the language of PL chain complexes, it was soon reformulated in terms of sheaf theory, and it was in this form that it quickly found much success, particularly in applications to algebraic geometry and representation theory. Early highlights in these directions include a key role in the proof of the Kazhdan–Lusztig conjecture, a singular variety version of the Weil conjectures, and generalizations to singular complex projective varieties of the "Kähler package" for smooth complex projective varieties, including a Lefschetz hyperplane theorem, a hard Lefschetz theorem, and Hodge decomposition and signature theorems[1]. In the time since, intersection homology has exploded. As of 2017, Mathematical Reviews records 700 entries that mention intersection homology or intersection cohomology, and this jumps to over 1100 when including the closely related perverse sheaves, which developed out of intersection homology. Viewpoints have also proliferated. In addition to definitions via PL and singular chains and through sheaf theory, an analytic L^2-cohomology formulation initially due to Jeff Cheeger developed concurrently to the work of Goresky and MacPherson, and another approach via what we might call perverse differential forms is the setting for some of the

[1] The book [140] by Kirwan and Woolf provides an excellent introduction to these applications of intersection homology.

most exciting current work in the field, providing a means to explore an intersection homology version of the rational homotopy type of a singular space. Each of these perspectives has its merits, and, as is often the case in mathematics, sometimes the most powerful results come by considering the interplay among different perspectives.

The intent of this book is to introduce the reader to the PL and singular chain perspectives on intersection homology. By this choice we do not mean at all to undervalue the other approaches. Rather, by sticking to the chain-theoretic context we hope to provide an introduction that will be readily accessible to the student or researcher familiar with the basics of algebraic topology without the need for the additional prerequisites of the sheaf-theoretic or more analytic formulations. This may then motivate the reader on to further study requiring more background; to facilitate this, we provide in Chapter 10 a collection of suggested references for the reader who wishes to pursue these other vantage points and their applications, including references for several excellent introductory textbooks and expositions. We also feel that the time is ripe for such a chain-based text given recent developments that allow for a thorough treatment of intersection homology duality via cup and cap products that completely parallels the modern approach to duality on manifolds as presented, for example, in Hatcher [125]. We provide such a textbook treatment for the first time here.

Prerequisites This book is intended to be as self-contained as possible, with the main prerequisite being a course in algebraic topology, particularly homology and cohomology through Poincaré duality. Some additional background in homological algebra may be useful throughout, and some familiarity with manifold theory and characteristic classes will serve as good motivation in the later chapters. In fact, we hope that this material might make for a good reading course for second- or third-year graduate students, as much of our development parallels and reinforces that of the standard tools of homology theory, though often the proofs need some modifications. The book also includes a number of sections, including the two appendices at the end, that provide some of the less standard background results in detail, as well as some expository sections regarding further directions and applications that there was not space to pursue here. When it is necessary to use facts from further afield, such as some occasional elementary sheaf theory or more advanced algebraic or geometric topology, we have attempted to provide copious references, with a preference for textbooks when at all possible. Our favored sources include topology texts by Hatcher [125], Munkres [181, 182], Dold [71], Spanier [220], Bredon [38], and Davis and Kirk [67]; books on PL topology by Hudson [130] and Rourke and Sanderson [198]; algebra books by Lang [147], Lam [146], and Bourbaki [30]; homological algebra books by Hilton and Stammbach [126], Weibel [238], and Rotman [197]; and introductions to sheaf theory by Bredon [37] and Swan [230].

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Many of the diagrams in this book were typeset using the TEX commutative diagrams package by Paul Taylor.

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Notations and Conventions

We now describe some conventions, notational and otherwise, we attempt to use throughout the book, making no claim to complete consistency.

Spaces

1. Manifolds and ∂-manifolds are usually denoted M or N. A manifold is a Hausdorff space that is locally homeomorphic to Euclidean space; we do not assume manifolds must be paracompact or second-countable. A ∂-manifold[2] is a Hausdorff space that is locally homeomorphic to Euclidean space or Euclidean half-space $\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1 \geq 0\}$; in other words, a ∂-manifold is what is often called a "manifold with boundary." The boundary of a ∂-manifold may be empty. "Manifold" will always mean a ∂-manifold with empty boundary. There is also an empty manifold of every dimension.
2. Arbitrary spaces have letters from the end of the alphabet such as Z, though sometimes also other letters. The space with one point is occasionally denoted pt.
3. Open subsets get letters such as U, V, W.
4. Subsets will be denoted $A \subset X$, rather than $A \subseteq X$; in other words $A \subset X$ includes the possibility that $A = X$. The interior of A is denoted $\mathring{A}$ and the closure is denoted $\bar{A}$.
5. Simplicial complexes will be given letters such as K, L. Subdivisions will generally be denoted by a prime, such as K'. We will often abuse notation and use K to represent both the simplicial complex (a space with a combinatorial structure as a union of simplices) and its underlying space as a topological space disregarding the extra structure. When we wish to emphasize the difference, for example in Appendix B, we will use $|K|$ to denote the underlying space.

[2] We mostly avoid the phrase "manifold with boundary," which sounds as though it specifies some particular class of manifolds but which is really a generalization of the concept of "manifold." Furthermore, we take the view that a "manifold with boundary" that has a non-empty boundary is not a manifold! This is because points on the boundary fail to satisfy the property that they should have Euclidean neighborhoods, which we take as part of the definition of being a manifold. The other problem is that "manifold with boundary" implies that there is a boundary and it is tempting to think then that the boundary cannot be empty. As an alternative, some authors have taken to using the notation "∂-manifold" as a replacement for "manifold with boundary." This seems to avoid these issues as well as eliminate some clunky phrasing.

6. When working with product spaces, we may write elements of $X \times Y$ as either (x, y) or $x \times y$. Product maps are usually written $f \times g$.
7. Generic maps between spaces will be denoted by letters such as f or g. The letter i, or variants such as i, generally denotes an inclusion. The map $\mathbf{d}$ is the diagonal map $\mathbf{d} : Z \to Z \times Z$, $\mathbf{d}(z) = (z, z)$.
8. While we will attempt to parenthesize fairly thoroughly, we will occasionally rely on a few simplifying conventions. In particular, expressions of the form $A - B$ should be understood as $(A) - (B)$. So, for example, $X \times Y - A \times B$ means $(X \times Y) - (A \times B)$ and not $X \times (Y - A) \times B$, and $X - K \cup L$ means $X - (K \cup L)$.
9. For a compact space Z, the space cZ is the open cone $cZ = [0, 1) \times Z/\sim$, where $\sim$ is the relation $(0, w) \sim (0, z)$ for all $w, z \in Z$. We typically denote the vertex of a cone by v. Similarly, the closed cone is $\bar{c}Z = [0, 1] \times Z/\sim$. More generally, for $r > 0$, we let $c_r Z = [0, r) \times Z/\sim$ and $\bar{c}_r Z = [0, r] \times Z/\sim$; in particular, $cZ = c_1 Z$. Then $c_r Z \subset \bar{c}_r Z \subset c_s Z \subset \bar{c}_s Z$ whenever $r < s$.
10. For a compact space Z, the (unreduced) suspension is $SZ = [-1, 1] \times X/\sim$, where the relation $\sim$ is such that $(-1, w) \sim (-1, z)$ and $(1, w) \sim (1, z)$ for any $w, z \in Z$. So $SZ = \bar{c}Z \cup_Z \bar{c}Z$.
11. When taking the product of a space with a Euclidean space, interval, or sphere, we usually put the Euclidean space, interval, or sphere on the left, e.g. $\mathbb{R} \times Z$ instead of $Z \times \mathbb{R}$. This has some ramifications for signs. For example, if ξ is a singular cycle in Z and $\bar{c}\xi$ denotes the singular cone on ξ in $\bar{c}Z$ (see Example 3.4.7), this is the convention that is consistent with adding the cone vertex as the first vertex and so gives us $\partial(\bar{c}\xi) = \xi$.
12. We use $\amalg$ to denote disjoint union.
13. Filtered spaces (our main object of study) are generally denoted by capital letters near the end of the alphabet, in particular X (or Y when we talk about multiple filtered spaces at the same time); the filtrations are usually left implicit in the sense that we say "the filtered space X." When we need to refer to the filtration explicitly, we let X^i denote the ith *skeleton* of the filtration, and we let $X_i = X^i - X^{i-1}$; see Section 2.2. The connected components of each $X^i - X^{i-1}$ are called *strata*. The *formal dimension* of a filtered space is generically denoted n (or m for a second filtered space). When we wish to emphasize the formal dimension of X, we write $X = X^n$. The *codimension* of X^i in X^n is $\operatorname{codim}(X^i) = n - i$. If S is a stratum in $X^i - X^{i-1}$, then $\operatorname{codim}(S) = \operatorname{codim}(X^i)$. Subspaces of filtered spaces, which inherit filtrations by intersection with the X^i, have letters like A or B, so we tend to have filtered pairs (X, A) or (Y, B).
14. If we wish to consider the underlying topological space of a filtered space X, i.e. we wish to explicitly disregard the filtration, we may write $|X|$.
15. The *singular locus* of a filtered space $X = X^n$ is defined to be X^{n-1} and can also be written Σ_X, or simply Σ if the space is clear. Strata contained in the singular locus are called *singular strata*.
16. Generic *strata* (see Section 2.2) of a filtered space have letters such as S and T. *Regular strata* are sometimes denoted R.

17. The *links* occurring in locally cone-like spaces (see Section 2.3), in particular CS sets or stratified pseudomanifolds, are denoted L or, occasionally, ℓ. We let $\mathrm{Lk}(x)$ denote the *polyhedral link* of a point in a piecewise linear space, i.e. if x is contained in the piecewise linear space X, then $\mathrm{Lk}(x)$ is the unique PL space such that x has a neighborhood piecewise linearly homeomorphic to $c\mathrm{Lk}(x)$; see [198, Section 1.1].
18. If X is a piecewise linear space, we let $\mathfrak{X}$ denote the filtered space with the underlying space of X but with its intrinsic PL filtration; see Section 2.10. Similarly, if X is a CS set, $\mathfrak{X}$ will denote the underlying space of X with its intrinsic filtration as a CS set.

Algebra

1. G will always be an abelian group, R a commutative ring with unity. In some contexts, R will be assumed to be a Dedekind domain, though this will be established at the relevant time.
2. Subgroups (or submodules) will be denoted $H \subset G$, rather than $H \subseteq G$; in other words $H \subset G$ includes the possibility that $H = G$.
3. We use the standard notations for standard algebraic objects: $\mathbb{Z}$ for integers, $\mathbb{Q}$ for rational numbers, $\mathbb{R}$ for real numbers (which also notates the *space* of real numbers, i.e. one-dimensional Euclidean space).
4. When working with R-modules in the context of a fixed ring R, we write $\mathrm{Hom}(A, B)$ and $A \otimes B$ rather than $\mathrm{Hom}_R(A, B)$ and $A \otimes_R B$.
5. Dedekind domains have cohomological dimension ≤ 1 (this follows from [197, Proposition 8.1] using that Dedekind domains are hereditary by definition [197, page 161]). Therefore, if R is a Dedekind domain, $\mathrm{Ext}^n_R(A, B) = 0$ for $n > 1$ and for any R-modules A, B. Hence, we simply write $\mathrm{Ext}(A, B)$ instead of $\mathrm{Ext}^1_R(A, B)$. Similarly, $\mathrm{Tor}^n_R(A, B) = 0$ for $n > 1$ and for any R-modules A, B; rather than $\mathrm{Tor}^1_R(A, B)$ we write $A * B$.
6. Generic purely algebraic chain complexes are denoted C_*, D_*, etc. Cohomologically graded complexes can be denoted C^*, D^*, etc.
7. For almost[3] all chain complexes, the boundary maps are all denoted ∂. For cohomologically graded complexes, we use d for the coboundary maps. If we wish to emphasize that ∂ is the boundary map of the chain complex C_*, we can write ∂_{C_*}, and analogously for coboundary maps of cochain complexes.
8. Elements of geometric chain complexes are typically denoted by lower-case Greek letters such as ξ, ζ, η, though we sometimes also use x, y, z. Note that we generally abuse notation by using the same symbol to refer to both a homology class and a chain representing it. For example, $\xi \in H_i(C_*)$ means that ξ is a homology class that we also think of as being represented by a cycle in C_i that we also denote ξ. In most contexts, this should not cause much confusion, though in those instances where confusion might reasonably occur, we use ξ just to denote the chain and $[\xi]$ to specify the homology class. We will indicate this notation specifically when it occurs. More generally, $[\cdot]$ indicates some

[3] We will see an exception in Section 6.2 for $I^{\bar{p}}S_*(X)$.

sort of equivalence class, so, depending on context, $[\xi]$ might reference a singular chain $\xi \in S_*(X)$ representing an element $[\xi] \in S_*(X, A)$ or an element $[\xi] \in H_*(X)$ or $[\xi] \in H_*(X, A)$. Similarly, if ξ is a simplicial chain, $[\xi]$ might denote the class in the PL chain complex $\mathfrak{C}_*(X)$ represented by ξ. See "Algebraic topology" notation item 5 below.

9. Elements of cochain complexes are denoted by lower-case Greek letters such as α, β, γ. Again, we typically abuse notation by using the same symbol to refer to both a cohomology class and a cochain representing it. For example, $\alpha \in H^i(C^*)$ means that α is a cohomology class that we also think of as being represented by a cocycle in C^i that we also denote α. In most contexts, this should not cause much confusion, though in those instances where confusion might reasonably occur, we use α just to denote the cochain and $[\alpha]$ to specify the cohomology class. We will indicate this notation specifically when it occurs. More generally, $[\cdot]$ indicates an equivalence class.
10. The connecting morphisms in long exact homology sequences are denoted ∂_*. The connecting morphisms in long exact cohomology sequences are denoted d^*.
11. Augmentation maps of chain complexes are denoted **a**; for example, we might have $\mathbf{a} : S_*(X) \to \mathbb{Z}$.
12. If x is an element of a chain or cochain complex, then we use $|x|$ to indicate the degree x. For example, if $x \in C_i$ or $X \in C^i$, then[4] $|x| = i$.

Algebraic topology

1. Δ^i denotes the standard geometric i-dimensional simplex. For definiteness, we can suppose that Δ^i is embedded in $\mathbb{R}^i$ with vertices

$$(0, \dots, 0), (1, 0, \dots, 0), \dots, (0, \dots, 0, 1).$$

By an "open simplex" or an "open face," we mean the interior of a simplex, e.g. the complement in Δ^i of the union of its faces of dimension $< i$.

2. Lower-case Greek letters such as σ, τ, and often others can denote either simplices in a simplicial complex or singular simplices, depending on context. The symbol $\mathring{\sigma}$ denotes an open simplex.
3. Lower-case Greek letters such as ξ and η will typically be used for chains and α and β will typically be used for cochains.
4. If ξ is a chain, then we use $|\xi|$ to indicate its support. If ξ is a simplicial chain, this is the union of the simplices appearing in ξ, while if ξ is singular it is the union of the images of the singular simplices of ξ. If σ is an oriented simplex in a simplicial complex, then we will typically write σ instead of $|\sigma|$ unless we really need to emphasize the notion of σ as a space. Note that $|\xi|$ might also indicate the degree of ξ, depending on context.

[4] Technically, this is not quite the right thing to do as the standard equivalence between homological and cohomological gradings tells us that the notation C^i should be equivalent to the notation C_{-i}. However, matters of degree will arise only when working with signs, and so $|x|$ will really only have significance mod 2. Therefore, we will live with this inconsistency.

5. Simplicial chain complexes are denoted $C_*(X)$, singular chain complexes are denoted $S_*(X)$, PL chain complexes are denoted $\mathfrak{C}_*(X)$. When there are subspaces or coefficients involved, the notations look like $C_*(X, A; G)$ for a subspace A and a coefficient group G. We use the same notation $H_*(X)$ for both homology groups $H_*(C_*(X))$ or $H_*(S_*(X))$, letting context determine which is meant. Since simplicial and PL chains often occur in the same context, we use $\mathfrak{H}_*(X)$ for $H_*(\mathfrak{C}_*(X))$.
6. If $f : X \to Y$ is a map of spaces, we abuse notation by letting f also denote both the induced chain maps of chain complexes defined on the spaces and the induced maps on homology, e.g. we write $f : S_*(X) \to S_*(Y)$ and $f : H_*(X) \to H_*(Y)$. The dualized maps of cochain complexes and cohomology groups are denoted f^*, e.g. $f^* : S^*(Y) \to S^*(X)$ and $f^* : H^*(Y) \to H^*(X)$. Similarly, if $f : C_* \to D_*$ is a purely algebraic map of chain complexes of R-modules, we also write $f : H_*(C_*) \to H_*(D_*)$ for the induced homology map and $f^* : H^*(\mathrm{Hom}(D_*, R)) \to H^*(\mathrm{Hom}(C_*, R))$ for the induced cohomology map.
7. For Mayer–Vietoris sequences, the map $H_*(U) \oplus H_*(V) \to H_*(U \cup V)$ will take (ξ, η) to $\xi + \eta$. Therefore, the map $H_*(U \cap V) \to H_*(U) \oplus H_*(V)$ will take ξ to $(\xi, -\xi)$.
8. The cross product chain map $S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$ (and its variants) can be written as either ε or $\times$. For example, we tend to write $\varepsilon : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$, but given two specific chains x, y, we may write $x \times y$. Unfortunately, it is common in algebraic topology to use the symbol $\times$ for both chain cross products and cochain cross products. We perpetuate this ambiguity, though context should make clear which is meant.
9. We use $\smile$ for cup products and $\frown$ for cap products. This distinguishes them from $\cup$ and $\cap$ for unions and intersections.
10. Fundamental classes are denoted Γ, with a decoration such as Γ_X if it is necessary to keep track of the space X.
11. The Poincaré duality map, consisting of a signed cap product with a fundamental class, is denoted $\mathcal{D}$.
12. We use $1 \in S^0(X)$ to denote the cocycle that evaluates to 1 on every 0-simplex. This is sometimes called the augmentation cocycle.

Intersection homology and cohomology

1. Perversities (see Section 3.1) are denoted $\bar{p}$, $\bar{q}$, $\bar{r}$, etc. In general, perversities will always have bars, with the exception[5] of the special perversities Q that occur in the discussion of the Künneth theorem; see Section 6.4.
2. $\bar{0}$ denotes the perversity that always evaluates to 0. We write $\bar{t}$ for the top perversity: $\bar{t}(S) = \mathrm{codim}(S) - 2$. By $\bar{m}$ and $\bar{n}$ we denote respectively the lower middle perversity

[5] This special case is partly historical, because there is little risk of confusion since Q is not used for anything else, and partly idiosyncratic. Probably we should use $\bar{Q}$.

and upper middle perversity, i.e.

$$\bar{m}(S) = \left\lfloor \frac{\operatorname{codim}(S) - 2}{2} \right\rfloor \qquad \text{(round down)},$$

$$\bar{n}(S) = \left\lceil \frac{\operatorname{codim}(S) - 2}{2} \right\rceil \qquad \text{(round up)}.$$

3. For a perversity $\bar{p}$, we let $D\bar{p}$ be the *dual* or *complementary* perversity with $D\bar{p}(S) = \bar{t}(S) - \bar{p}(S)$ for all singular strata S; see Definition 3.1.7.
4. Throughout the first part of the book, simplicial, PL, and singular perversity $\bar{p}$ intersection chain complexes are written $I^{\bar{p}}C_*^{\mathrm{GM}}(X)$, $I^{\bar{p}}\mathfrak{C}_*^{\mathrm{GM}}(X)$, $I^{\bar{p}}S_*^{\mathrm{GM}}(X)$, with corresponding homology groups $I^{\bar{p}}H_*^{\mathrm{GM}}(X)$, $I^{\bar{p}}\mathfrak{H}_*^{\mathrm{GM}}(X)$, $I^{\bar{p}}H_*^{\mathrm{GM}}(X)$. The GM here stands for "Goresky–MacPherson." In Chapter 6, we introduce the variant "non-GM" intersection homology and the notation becomes simply $I^{\bar{p}}C_*(X)$, $I^{\bar{p}}\mathfrak{C}_*(X)$, and $I^{\bar{p}}S_*(X)$ with corresponding homology groups $I^{\bar{p}}H_*(X)$, $I^{\bar{p}}\mathfrak{H}_*(X)$, and $I^{\bar{p}}H_*(X)$.
5. When we introduce non-GM intersection homology, the definition will use a modified boundary map that we write as $\hat{\partial}$; see Section 6.2.1.
6. For intersection cohomology, we raise the index and lower the perversity marking, e.g. $I_{\bar{p}}S^*(X)$ and $I_{\bar{p}}H^*(X)$. Lowering the perversity symbol has no intrinsic meaning; it is meant as a further distinguishing aid between homology and cohomology.
7. We write the intersection product, which appears primarily in Section 8.5, with the symbol $\pitchfork$. Note that this differs from the use of this symbol in the early intersection homology literature, such as [105], where $A \pitchfork B$ typically means A and B are in (stratified) general position. In [105], the intersection product is written with $\cap$, but for us this risks confusion with the cap product. In other sources the intersection product of chains is sometimes written $\xi \bullet \eta$ or $\xi \cdot \eta$. We prefer to utilize $\pitchfork$ as the intersection pairing and to state transversality in words.

Miscellaneous conventions

Signs

1. We utilize throughout the Koszul sign conventions, so that interchange of elements of degrees i and j usually results in a sign $(-1)^{ij}$. See the Appendix A.1 for details.
2. The standard exception to the Koszul rule, necessary for evaluation to be a chain map, is that the sign occurring in the coboundary map of the chain complex $E^* = \operatorname{Hom}^*(C_*, D_*)$ has the form

$$(d_E^* f)(c) = \partial_{D_*}(f(c)) - (-1)^{|f|} f(\partial_{C_*}(c))$$

for $c \in C_*$ and $f \in \operatorname{Hom}^*(C_*, D_*)$. In particular, if $\alpha \in \operatorname{Hom}^i(C_*, R) = \operatorname{Hom}(C_i, R)$, then $df = (-1)^{i+1} f\partial$.
3. The connecting morphisms of long exact homology sequences have degree -1 and so can generate signs upon interchanges.

Identity map

The expression id is used for the identity function. It can be either a topological or algebraic identity. Context will usually make clear which identity function is meant, though we can make it precise with subscripts such as $\mathrm{id}_X : X \to X$ or $\mathrm{id}_{C_*} : C_* \to C_*$.

Parentheses

1. When a function f acts on an element x of a set, group, etc., we generally write $f(x)$. The standard exception will be boundary maps ∂ acting on a chain ξ, which we will usually write as $\partial\xi$.
2. To avoid the ambiguity inherent in writing expressions such as $\partial\xi \otimes \eta$, we will write either $\partial(\xi \otimes \eta)$ or $(\partial\xi) \otimes \eta$, as appropriate. We also use $\xi \otimes \partial\eta$, as there is no ambiguity here.
3. When parentheses are omitted, expressions compile from the right. For example, if $f : X \to Y$ and $g : Y \to Z$, then, as usual, $gf(x)$ means $g(f(x))$. As a more complex example, $\Phi(\mathrm{id} \otimes \beta)\partial(\xi \otimes \eta)$ means $\Phi((\mathrm{id} \otimes \beta)(\partial(\xi \otimes \eta)))$.
4. We will use an obnoxious number of parentheses to describe spaces as clearly as possible. As noted in “Spaces” notation, item 8 above, one place where we will sometimes avoid this is when considering complements.