1
Introduction

Let us begin with some motivation, followed by some general remarks about the structure of this book and what can be found (and not found!) in it.

1.1 What Is Intersection Homology?

Perhaps the most significant result about the topology of manifolds is the Poincaré Duality Theorem: If M is a closed connected oriented n-dimensional manifold and $\Gamma \in H_n(M) \cong \mathbb{Z}$ is a generator, then the cap product $\frown \Gamma : H^i(M) \to H_{n-i}(M)$ is an isomorphism for all i. There are more general versions with more bells and whistles, but, in any form, Poincaré duality, and related invariants such as signatures and L-classes, is a fundamental tool in the study and classification of manifolds.

Unfortunately, Poincaré duality fails in general for spaces that are not manifolds. In fact, it is enough for a space to have just one point that is not locally Euclidean. For example, let $S^n \vee S^n$ be the one-point union of two n-dimensional spheres, $n > 0$. Then $H_0(S^n \vee S^n) \cong \mathbb{Z}$ but $H^n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$. Or, as a slightly more substantive example, one where we cannot simply pull the two pieces apart, consider the suspended torus ST^2 (Figure 1.1). This three-dimensional space has two "singular points," each of which has a neighborhood homeomorphic to the cone on the torus cT^2, and the cone point of cT^2 does not have a neighborhood homeomorphic to $\mathbb{R}^3$. Perhaps the easiest way to show this also illustrates the power of algebraic topology: If we let v be the cone point of cT^2, then, as cones are contractible, the long exact sequence of the pair and homotopy invariance of homology give us

$$H_2(cT^2, cT^2 - \{v\}) \cong H_1(cT^2 - \{v\}) \cong H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

But if v has a neighborhood homeomorphic to $\mathbb{R}^3$, then by excision we would have

$$H_2(cT^2, cT^2 - \{v\}) \cong H_2(\mathbb{R}^3, \mathbb{R}^3 - \{v\}) \cong H_1(\mathbb{R}^3 - \{v\}) \cong H_1(S^2) \cong \mathbb{Z}.$$

So ST^2 is not a manifold, and routine computations show that

$$\begin{aligned} H_3(ST^2) &= \mathbb{Z}, & H^3(ST^2) &= \mathbb{Z}, \\ H_2(ST^2) &= \mathbb{Z} \oplus \mathbb{Z}, & H^2(ST^2) &= \mathbb{Z} \oplus \mathbb{Z}, \\ H_1(ST^2) &= 0, & H^1(ST^2) &= 0, \\ H_0(ST^2) &= \mathbb{Z}, & H^0(ST^2) &= \mathbb{Z}. \end{aligned}$$

So, for example, $H_2(ST^2) \not\cong H^1(ST^2)$. Poincaré duality fails.

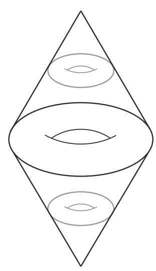

Figure 1.1 The suspended torus ST^2.

But spaces with *singularities*, points that do not have Euclidean neighborhoods, are both important and not always all that pathological. Many of them, such as our suspension example, possess dense open subsets that are manifolds. For example, if we remove the two suspension points from ST^2 we have $(0,1) \times T^2$, a manifold. Much more significant classes of examples come by considering algebraic varieties and orbit spaces of manifolds and varieties by group actions. In general such spaces may have singularities, and they will not necessarily just be isolated points. But with some reasonable assumptions (for example, assuming the group actions are nice enough or that the varieties are complex irreducible – see Section 2.8), such spaces will contain dense open manifold subsets, and, in fact, they will be filtered by closed subsets

$$X = X^n \supset X^{n-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$$

in such a way that each $X^k - X^{k-1}$ will be a manifold or empty. Such filtrations of spaces may be in some way intrinsic to the space (Figure 1.2), or they may be imposed by some other consideration such as the desire to study a manifold together with embedded subspaces (Figure 1.3).

The connected components of the $X^k - X^{k-1}$ are called *strata*. When $k < n$, we say they are *singular strata*, even though, depending on the choice of stratification, they may contain points with Euclidean neighborhoods. The subspace X^{n-1}, which is the union of the singular strata, is also called the *singular locus* or *singular set* and denoted Σ. The components of $X^n - X^{n-1} = X - \Sigma$ are called *regular strata*. It is usually too much to ask

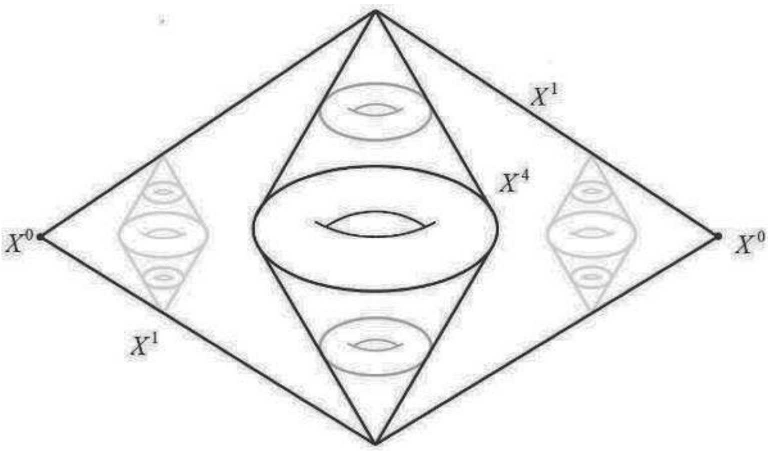

Figure 1.2 The twice suspended torus $X = S(ST^2)$. This space has a natural filtration in which X^0 comprises the suspension points of the second suspension, $X^1 = X^2 = X^3$ is the suspension of the suspension points of the first suspension, and $X^4 = X$. Note that X^0 is a 0-manifold, $X^1 - X^0$ is two open intervals, $X^2 - X^1 = X^3 - X^2 = \emptyset$, and $X^4 - X^3 \cong (-1, 1) \times (-1, 1) \times T^2$.

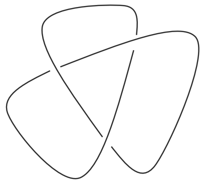

Figure 1.3 A manifold embedded in the ambient manifold S^3 (not shown).

for something like a tubular neighborhood around a singular stratum, i.e. a neighborhood homeomorphic to a fiber bundle, but, perhaps again with some additional conditions, the "normal behavior" along singular strata will be locally uniform. A typical condition is that a point $x \in X^k - X^{k-1}$ should have a neighborhood U of the form $U \cong \mathbb{R}^k \times cL$, where L is a compact filtered space and such that the homeomorphism takes $\mathbb{R}^k \times \{v\}$, again letting v be the cone point, to a neighborhood of x in $X^k - X^{k-1}$. For the remainder of this introductory discussion, we will limit ourselves to discussing the class of stratified spaces called *(stratified) pseudomanifolds*, defined formally in Section 2.4, which possess all of these nice local properties and which is a broad enough class to encompass all irreducible complex analytic varieties and all connected orbit spaces of smooth actions of compact Lie groups on manifolds. For simplicity of discussion, we also assume through this introduction that all spaces are compact, connected, and oriented.

Given all the manifold structure present and the other good behaviors of such spaces, it is reasonable to ask whether there might be some way to recover some version of Poincaré duality after all. This is precisely what Mark Goresky and Robert MacPherson did in [105] by introducing *intersection homology*. Intersection homology is defined by modifying the definition of the homology groups $H_*(X)$ so that only chains satisfying certain extra geometric conditions are allowed. These geometric conditions are governed by a *perversity parameter* $\bar{p}$, which assigns an integer to each singular stratum of the space. The result is the perversity $\bar{p}$ intersection chains $I^{\bar{p}}C_*(X)$ and their homology groups $I^{\bar{p}}H_*(X)$. Furthermore, to each perversity $\bar{p}$ there is a complementary *dual perversity* $D\bar{p}$, and Goresky and MacPherson showed that, given certain assumptions on X and $\bar{p}$, there are *intersection pairings*

$$I^{\bar{p}}H_i(X) \otimes I^{D\bar{p}}H_{n-i}(X) \to \mathbb{Z}$$

that become nonsingular over the rationals, i.e. after tensoring everything with $\mathbb{Q}$.

Let us provide a rough sketch of the basic idea of how and why this all works. We will be very loose about the specific details here, but more about this material and the original construction of the Goresky–MacPherson intersection pairing can be found in Section 8.5 and, of course, in [105].

To get at the idea, we must first ask what it is that makes manifolds so special. One consequence of their locally Euclidean nature is that it is possible to take advantage of general position: If M^n is a smooth manifold and P^p and Q^q are two smooth submanifolds, then it is possible to perturb one of P or Q so that the intersection $P \cap Q$ will be a manifold of dimension $p + q - n$. In particular, we can find a Euclidean neighborhood U_x of any point $x \in P \cap Q$ so that the triple $(U, P \cap U, Q \cap U)$ is homeomorphic to the triple $(\mathbb{R}^n, \mathbb{R}^p \times \{0\}, \{0\} \times \mathbb{R}^q)$, with the intersection of the two subspaces having dimension $p+q-n$ and providing a Euclidean neighborhood of x in $P \cap Q$; see e.g. [38, Section II.15]. Furthermore, if M, P, and Q are all oriented, it is possible to orient $P \cap Q$ by a construction involving bases for these local vector spaces [38, Section VI.11.12][1]. These ideas can be extended so that, if ξ and η are two chains in M (simplicial, piecewise linear, or singular) that satisfy an appropriate notion of general position, then there is defined an intersection $\xi \pitchfork \eta$ of degree $\deg(\xi) + \deg(\eta) - n$. This notion yields a partially-defined product on chains $\pitchfork\colon C_i(M) \otimes C_j(M) \to C_{i+j-n}(M)$. It is not fully defined because we cannot meaningfully intersect chains that are not in general position, just as the intersection of two submanifolds not in general position will not generally be a manifold. However, this intersection pairing is well defined as a map $\pitchfork\colon H_i(M) \otimes H_j(M) \to H_{i+j-n}(M)$ because any two cycles can be pushed into general position without changing their homology classes, and the homology class of the resulting intersection does not depend on the choices. Of particular note are the products $\pitchfork\colon H_i(M) \otimes H_{n-i}(M) \to H_0(M)$ because composing with the augmentation map $\mathbf{a}$ then yields a bilinear pairing $\pitchfork\colon H_i(M) \otimes H_{n-i}(M) \to H_0(M) \xrightarrow{\mathbf{a}} \mathbb{Z}$. As any homomorphism to $\mathbb{Z}$ must take any element of finite order to 0, this intersection pairing

[1] Technically, what we have described here is *transversality*, while *general position* is simply the requirement in an n-manifold that a p-manifold and a q-manifold meet in a subspace of dimension $\leq p + q - n$.

descends to a map $H_i(M)/T_i(M) \otimes H_{n-i}(M)/T_{n-i}(M) \to \mathbb{Z}$, where we let $T_*(M)$ denote the torsion subgroup of $H_*(M)$.

What does this have to do with Poincaré duality? If M is a closed oriented n-manifold, then Poincaré duality and the Universal Coefficient Theorem together yield isomorphisms

$$H_i(M) \cong H^{n-i}(M) \cong \operatorname{Hom}(H_{n-i}(M), \mathbb{Z}) \oplus \operatorname{Ext}(H_{n-i-1}(M), \mathbb{Z}).$$

Some elementary homological algebra then allows us to derive from this an isomorphism

$$H_i(M)/T_i(M) \cong \operatorname{Hom}(H_{n-i}(M)/T_{n-i}(M), \mathbb{Z}).$$

Some slightly more elaborate homological algebra also leads to an isomorphism

$$T_i(M) \cong \operatorname{Hom}(T_{n-i-1}(M), \mathbb{Q}/\mathbb{Z}).$$

Applying the adjunction relation, these two isomorphisms can be interpreted as nonsingular bilinear pairings

$$H_i(M)/T_i(M) \otimes H_{n-i}(M)/T_{n-i}(M) \to \mathbb{Z},$$
$$T_i(M) \otimes T_{n-i-1}(M) \to \mathbb{Q}/\mathbb{Z}.$$

The first of these turns out to be precisely the intersection pairing! And the second is the closely related torsion linking pairing. If $\xi \in C_i(M)$ is a cycle with $k\xi = \partial\zeta$ for some $k \in \mathbb{Z}$, $k \neq 0$, then the linking pairing of $\xi \in T_i(M)$ with $\eta \in T_{n-i-1}(M)$ can be computed as $1/k$ times the intersection number of ζ with η, assuming the chains are all in general position. This number is well defined in $\mathbb{Q}/\mathbb{Z}$.

Prior to the invention of the modern version of cohomology, Poincaré duality was formulated in these terms. These days, most readers will be more familiar with the nonsingular cup product pairing $H^i(M)/T^i(M) \otimes H^{n-i}(M)/T^{n-i}(M) \xrightarrow{\smile} \mathbb{Z}$, which turns out to be isomorphic to the intersection pairing via the Poincaré duality isomorphisms. In general, cup products are simpler to define than intersection products; they are defined at the cochain level $C^i(M) \otimes C^j(M) \xrightarrow{\smile} C^{i+j}(M)$, and, perhaps most importantly, the cup product can be defined on any space, though in general we do not obtain a nonsingular pairing. The only downside to the cup product is that it obfuscates this beautiful geometric interpretation of Poincaré duality, an interpretation that will allow us to see clearly what goes wrong for spaces that are not manifolds.

So, let us return to spaces with singularities. As a simple example, consider $X = M_1 \vee M_2$, the wedge of two n-manifolds, $n > 2$. In a manifold of dimension $n > 2$, any two curves can be perturbed to be disjoint as $1 + 1 - n < 0$. But in $X = M_1 \vee M_2$, any two curves that pass through the wedge point v cannot be separated (unless one only intersects $\{v\}$ at an endpoint). Furthermore, even if $n = 2$ and ξ and η are two 1-chains that have an isolated intersection at v, the lack of a local Euclidean neighborhood makes it unclear how to orient the intersection point, which is a necessary step in defining an intersection product (Figure 1.4). So we see that singularities are not compatible with having well-defined intersection products.

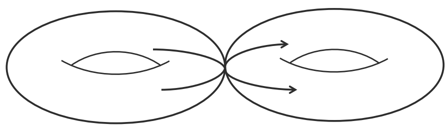

Figure 1.4 A failure of general position. What's the intersection number of the two curves depicted?

Or are they? The fundamental insight of Goresky and MacPherson was that if chains don't intersect well at singularities, perhaps they shouldn't be allowed to interact with the singularities too much. In fact, roughly stated, the allowability condition that a chain ξ must satisfy to be a perversity $\bar{p}$ intersection chain says that, if S is any singular stratum of an n-dimensional space X and if S has dimension k and ξ is an i-chain with support $|\xi|$, then

$$\dim(|\xi| \cap S) \le i + k - n + \bar{p}(S), \tag{1.1}$$

and a similar condition must hold for $\partial\xi$. There is a way to make this precise with singular chains, but for now the reader will be safe imagining simplicial chains to make better sense of these dimension requirements. Without the $\bar{p}(S)$ summand, inequality (1.1) would be precisely the requirement that $|\xi|$ and S be in general position if X were a manifold. The $\bar{p}(S)$ term allows for some deviation from the strict general position formula; hence *perversity*. The complex of chains satisfying these conditions is the *perversity $\bar{p}$ intersection chain complex $I^{\bar{p}}C_*(X)$*, and the resulting homology groups $I^{\bar{p}}H_*(X)$ are the *perversity $\bar{p}$ intersection homology groups*.

Now suppose ξ is a $\bar{p}$-allowable i-chain, i.e. $\xi \in I^{\bar{p}}C_i(X)$, and that η is a $\bar{q}$-allowable j-chain. We will also suppose that there is a perversity $\bar{r}$ such that for each singular stratum S we have $\bar{p}(S) + \bar{q}(S) \le \bar{r}(S) \le \bar{t}(S)$, where $\bar{t}$ is the *top perversity* defined by $\bar{t}(S) = \operatorname{codim}(S) - 2 = \dim(X) - \dim(S) - 2$. Lastly, we suppose that our space X is a *stratified pseudomanifold* and that ξ and η are in *stratified general position*, which means that they should satisfy the general position inequality within each singular stratum:

$$\dim(S \cap |\xi| \cap |\eta|) \le \dim(S \cap |\xi|) + \dim(S \cap |\eta|) - \dim(S).$$

With these assumptions, it is possible to define an intersection $\xi \pitchfork \eta$ that is an $\bar{r}$-allowable $i + j - n$ chain! Furthermore, work of Clint McCrory [170, 171] shows that it is possible to push any $\bar{p}$-allowable cycle ξ and $\bar{q}$-allowable cycle η into stratified general position and in such a way that the resulting homologies between cycles also satisfy the respective allowability conditions. We therefore arrive at a map

$$\pitchfork\colon I^{\bar{p}}H_i(X) \otimes I^{\bar{q}}H_j(X) \to I^{\bar{r}}H_{i+j-n}(X),$$

generalizing the intersection product for manifolds. If $\bar{q}$ is the *complementary perversity* $D\bar{p}$, which is defined so that $\bar{p}(S) + D\bar{p}(S) = \bar{t}(S) = \operatorname{codim}(S) - 2$, then by composing with an augmentation map we get a pairing

$$I^{\bar{p}}H_i(X) \otimes I^{D\bar{p}}H_{n-i}(X) \xrightarrow{\pitchfork} I^{\bar{t}}H_0(X) \xrightarrow{\mathbf{a}} \mathbb{Z}.$$

The intersection homology Poincaré Duality Theorem of [105] says that this pairing becomes nonsingular when tensored with $\mathbb{Q}$. If M is a manifold (unstratified), the perversity conditions become vacuous, and this pairing reduces to the intersection pairing over $\mathbb{Z}$, which is nonsingular when tensored with $\mathbb{Q}$.

To give an idea about why this pairing works after having argued that intersection pairings are not so compatible with singularities, notice that if $\xi \pitchfork \eta$ is a $\bar{t}$-allowable 0-chain then its intersection with the singular stratum S must satisfy

$$\dim(|\xi \pitchfork \eta| \cap S) \leq 0 - \operatorname{codim}(S) + \bar{t}(S) = -2.$$

So, in other words, $|\xi \pitchfork \eta|$ must be contained in the dense manifold part of X. In fact, with a bit more work, the allowability and stratified general position conditions imply that $|\xi| \cap |\eta| \subset X - \Sigma$, the dense submanifold of X. So the bad behavior discussed previously cannot happen because the intersection of chains of complementary dimension and complementary perversity is forced to happen in the nice manifold portion of the space, not at the singularities. If ξ and η do not have complementary dimensions, it is possible that $|\xi| \cap |\eta|$ might have a nontrivial intersection with Σ, but the $\bar{r}$-allowability of $\xi \pitchfork \eta$ shows that such intersections within the singular locus are carefully controlled by the perversity data.

Here is another important motivating example that provides some idea of why intersection homology Poincaré duality might work out. Let M be a compact oriented n-dimensional manifold with boundary $\partial M \neq \emptyset$. Let $X = M/\partial M$. So we can think of X as M with its boundary collapsed to a point or, up to homeomorphism, it is M with the closed cone $\bar{c}(\partial M)$ adjoined, $X \cong M \cup_{\partial M} \bar{c}(\partial M)$. If we let v be the cone point, then v will not in general have a Euclidean neighborhood unless, for example, $\partial M \cong S^{n-1}$. So it is natural to stratify X by $\{v\} \subset X$, and any perversity on X is determined by the single value $p := \bar{p}(\{v\})$. Without working carefully through the details here, the basic idea is that if i is small compared to a value depending on p, then the allowability condition (1.1) will prevent i-chains in $I^{\bar{p}}C_i(X)$ from intersecting v. So the low-dimensional chains behave as though the cone point is not there, and we get $I^{\bar{p}}H_i(X) \cong H_i(X - \{v\}) \cong H_i(M)$. On the other hand, if i is large enough then the allowability condition will be satisfied for any i-chain, noting that $\dim(|\xi| \cap \{v\}) \leq 0$ because v is a point, and so all i-chains can be utilized. Therefore, $I^{\bar{p}}H_i(X) \cong H_i(X)$, and so $I^{\bar{p}}H_i(X) \cong H_i(M, \partial M)$ if $i > 0$. It turns out that there is only one middle dimension in which there is a transition between these behaviors, and in that dimension we get $I^{\bar{p}}H_i(X) \cong \operatorname{im}(H_i(M) \to H_i(M, \partial M))$. Altogether, the precise statement works out as follows, assuming $p < n - 1$:

$$I^{\bar{p}}H_i(X) \cong \begin{cases} H_i(M, \partial M), & i > n - p - 1, \\ \operatorname{im}(H_i(M) \to H_i(M, \partial M)), & i = n - p - 1, \\ H_i(M), & i < n - p - 1. \end{cases}$$

But now recall that the Lefschetz Duality Theorem for manifolds with boundary provides a duality isomorphism $\frown \Gamma : H^i(M) \to H_{n-i}(M, \partial M)$, and, modulo torsion, this can also be partially interpreted in terms of a nonsingular intersection pairing $H_i(M) \otimes H_{n-i}(M, \partial M) \to H_0(M) \to \mathbb{Z}$, with the geometric intersections occurring in the interior of M. Lefschetz duality also implies a nondegenerate intersection pairing among the groups $\operatorname{im}(H_i(M) \to H_i(M, \partial M))$; see Section 8.4.5 for more details. As we vary the perversity, intersection homology of X provides all of these groups! And the duality between the perversity $\bar{p}$ and its dual $D\bar{p}$ positions the behavioral transitions in complementary dimensions: Notice that the dual $D\bar{p}$ takes the value $D\bar{p}(\{v\}) = n-2-p$, so indeed $(n-p-1)+(n-(n-2-p)-1) = n$. So the intersection homology pairings generate the Lefschetz duality pairings as special cases!

One seeming deficiency in the intersection homology groups is that the intersection pairing $I^{\bar{p}}H_i(X) \otimes I^{D\bar{p}}H_{n-i}(X) \to \mathbb{Z}$ is not just between complementary dimensions but between complementary perversities. So even when $n = 2k$, we do not necessarily have a middle-dimensional pairing of a group with itself. In manifold theory, if $n = 2k$ then such self-pairings $H_k(M) \otimes H_k(M) \to \mathbb{Z}$ are symmetric for k even and antisymmetric for k odd, and such pairings possess their own algebraic invariants, such as the signature for k even, that play a key role in manifold classification. Given a version of Poincaré duality for stratified spaces, such invariants are the desired consequence. In general, however, there is no self-complementary perversity such that $\bar{p} = D\bar{p}$. However, there are two dual perversities, $\bar{m}$ and $\bar{n} = D\bar{m}$, called the lower and upper middle perversities, and these are as close as possible. If the pseudomanifold X satisfies certain local intersection homology vanishing conditions, then $I^{\bar{m}}H_*(X)$ and $I^{\bar{n}}H_*(X)$ will be isomorphic and we do get a self-pairing. Already in [105], Goresky and MacPherson observed that this is the case for spaces stratifiable by strata only of even codimension, and this includes complex varieties. Important broader classes of such spaces were introduced later, including Witt spaces by Paul Siegel [218] and IP spaces(intersection homology Poincaré spaces) by William Pardon [187]. As is the signature for manifolds, the intersection homology signature (and, in fact, a more refined invariant – the class of the intersection pairing in the Witt group) is a bordism invariant of such spaces, and this has ramifications toward the geometric representation of certain generalized homology theories, including ko-homology and $\mathbb{L}$-homology, by bordisms of stratified spaces. This fact can also be used to construct for such spaces a version of the characteristic L-classes in ordinary homology. We provide an exploration of these topics in our culminating chapter, Chapter 9.

Another seeming shortcoming of intersection homology duality is that the intersection pairing is in general only nonsingular after tensoring with $\mathbb{Q}$. Over $\mathbb{Z}$, the map $I^{\bar{p}}H_i(X) \to \operatorname{Hom}(I^{D\bar{p}}H_{n-i}(X), \mathbb{Z})$ adjoint to the intersection pairing is injective, making the pairing nondegenerate, but it is not necessarily an isomorphism and so the pairing is not necessarily nonsingular. But, in fact, this must be the most we can hope for in general, as the intersection pairing on the groups $\operatorname{im}(H_i(M) \to H_i(M, \partial M))$ for a manifold only need be nondegenerate, not necessarily nonsingular, and we have already seen that this occurs as a special case of intersection homology duality[2]. Yet there are local "torsion-free" conditions due to Goresky and Siegel [111] that can be imposed on a space to imply nonsingularity of the pairing over $\mathbb{Z}$, as well as the existence of nonsingular torsion linking pairings analogous to those for manifolds. More recent work on such spaces has developed intersection cohomology and cup and cap products, so that now intersection Poincaré duality can also be expressed as an isomorphism of the form $\frown \Gamma : I_{\bar{p}}H^i(X) \to I^{D\bar{p}}H_{n-i}(X)$. This formulation was introduced with field coefficients in [100], for which the torsion-free conditions are automatic, and is developed here in Chapters 7 and 8 over more general rings, including $\mathbb{Z}$.

1.2 Simplicial vs. PL vs. Singular

As the reader should be aware from an introductory algebraic topology course, there are several ways to define homology groups on a space, and, assuming the space is nice enough, those definitions that the space admits will yield isomorphic homology groups. Each such definition has its own advantages and disadvantages: homology via CW complexes is difficult to set up technically but then often allows for the simplest computations; simplicial homology is defined combinatorially and very amenable to computations by computer but enforces a somewhat rigid structure on spaces that can make it difficult to prove theorems or work with subspaces; singular homology is defined on arbitrary spaces and is often the best setting to prove theorems but it is usually hopeless for direct computation from the definitions. We encounter the same trade-offs in intersection homology. CW homology is not really available at all, and so we have simplicial and singular homology, each of which will be treated in this book.

There is yet another species of homology we will utilize that occupies something of a middle ground between simplicial and singular homology: piecewise linear (or PL) homology. The basic idea is that PL chains are linear combinations of geometric simplices, just like in simplicial homology, but the simplices are not required to all come from the same triangulation. Technically, a PL chain lives in the direct limit of simplicial chain complexes, with the limit being taken over all suitably compatible triangulations of the space and with the maps in the direct system being induced by geometric subdivision of triangulations. We will see in Theorem 3.3.20 that for any PL filtered space the PL intersection homology groups (and hence ordinary PL homology groups on any PL space) are isomorphic to the simplicial groups with respect to any triangulation satisfying a mild hypothesis (that the triangulation be full). With some other mild assumptions, we will show that these simplicial and PL intersection homology groups are isomorphic to the singular intersection homology groups in Theorem 5.4.2. As we do not have an acyclic carrier theorem available in intersection homology, it would be much more difficult to establish an isomorphism between simplicial and singular intersection homology without using the PL theory as an intermediary.

[2] It's not a bug, it's a feature!

One of the advantages that PL homology has over simplicial homology is that it behaves much better with respect to the consideration of open subsets. An open subset of a triangulated space needs to be given its own triangulation in order to speak of its simplicial chains; but as PL homology already considers all triangulations, a PL chain in X that is supported in an open subset U is already a PL chain in U without worrying about the specific triangulation. Consequently, we obtain excision and Mayer–Vietoris theorems for PL homology that mirror the singular homology theorems and so are more general than what one sees for simplicial homology. Another technical advantage, which we shall only touch upon lightly in Section 8.5, is that PL chains provide a good setting for defining intersection pairings, which Goresky and MacPherson used to demonstrate Poincaré duality for intersection homology when they introduced it in [105].

As we progress to the later stages of the book, however, the technical advantages of the PL approach will begin to lessen as the technical difficulties begin to escalate. For example, as the cup and cap products in intersection homology cannot be defined using an Alexander–Whitney-type formula (as far as we know), the simplicial approach does not provide any utility toward computing these products. At the same time, the direct limit that arises in the definition of PL chains dualizes to an inverse limit for PL cochains, and these can be difficult to work with. Consequently, when we reach intersection cohomology, we will discuss briefly the PL intersection cohomology groups, but we will limit our discussion of products and duality to the singular chain setting.

1.3 A Note about Sheaves and Their Scarcity

As documented by Steven Kleiman in his somewhat controversial historical survey of the early development of intersection homology [141], after first developing PL intersection homology Goresky and MacPherson soon discovered that their work dovetailed with research in algebraic geometry that Pierre Deligne was undertaking from the point of view of sheaf theory[3]. Goresky and MacPherson quickly recognized the power of the tools available working in the derived category of complexes of sheaves on a space, especially a sheaf-theoretic duality theorem called Verdier duality, and intersection homology was reformulated in these terms in [106]. Using Verdier duality, they provided a proof of intersection homology duality on topological pseudomanifolds, extending their duality results beyond the piecewise linear pseudomanifolds of [105]. Furthermore, sheaf theory provides a good axiomatic framework, which enabled them in [106] to prove that, with certain restrictions on $\bar{p}$, the groups $I^{\bar{p}}H_*(X)$ are topological invariants; in other words they do not depend on the choice of stratification.

From here, the sheaf-theoretic perspective on intersection homology largely took over, and it has been the venue of many of the most significant applications of intersection

[3] At the same time, Jeff Cheeger was developing a similar theory from the analytic point of view using L^2-cohomology [59, 61].