1 Basic Equations for Long Waves

This chapter presents the derivation of the basic equations that we will use in analyses and calculations of unsteady free-surface flows in natural or man-made channels, e.g. tidal or fluvial channels, shipping canals and irrigation canals. We deal with a mass balance and a momentum balance integrated across the entire flow cross section, assuming a hydrostatic pressure distribution.

1.1 Approach

The principal subject of this book is the class of unsteady free-surface flows of water with a characteristic length scale that is far greater than the depth, the so-called long waves. Tides, storm surges and flood waves in rivers provide good examples of this category (contrary to ship waves or wind-generated waves, whose lengths are usually not large or even small compared with the depth).

We restrict ourselves to flows in relatively narrow, weakly curved conduits such as tidal channels and rivers, in which the main flow direction is determined by the geometry of the boundary, which is assumed to be given beforehand (excluding morphological changes such as meandering of rivers). In these cases, the bulk flow direction is known so that only the flow intensity (the discharge, say) is to be determined, in addition to the water surface elevation. A typical area of application, a long, slowly winding river reach with lateral side basins, is shown in Figure 1.1.

As expressed by their name, long waves are characterized by length dimensions that far exceed the depths. This implies that the curvature of the streamlines in the vertical plane is negligible, for which reason we will assume a *hydrostatic pressure distribution* in the vertical. This greatly simplifies the schematization and the calculations.

In bends, the flow is forced to change direction through a lateral variation of the water level, being higher at the outer bank and lower at the inner bank. This is essential in detailed computations of the spiral flow in bends, but it is irrelevant for the large-scale computations of longitudinal variations with which we are concerned. So we will ignore lateral variations in surface elevation. The height of this level above the adopted reference plane $z = 0$ is designated as h. This quantity is a function of the downstream coordinate s (measured along the axis of the conduit) and the time t, or $h = h(s, t)$.

Because the water level is assumed not to vary within the cross section, the same applies to the downstream pressure gradient driving the flow. Therefore, instead of working with the point values of the velocities within each cross section, it is feasible to work with

Fig. 1.1 Reach of the river Rhine in The Netherlands (Pannerdens Kanaal) with groyne fields and laterally connected side basins; from, https://beeldbank.rws.nl, Rijkswaterstaat / Bart van Eyck

cross-sectionally integrated flow velocities (i.e. the total volume flux or flow rate, hereafter referred to as the discharge Q, also for purely oscillatory flows).

Summarizing, we have two dependent variables (h and Q) that have to be determined as functions of the longitudinal coordinate and time:

$$h = h(s,t) \quad \text{and} \quad Q = Q(s,t)$$

This requires a so-called *one-dimensional flow model*, typified by the dependence on only one space coordinate.

This chapter presents the derivation of the basic equations that we will use in the analyses and calculations in the following chapters. We deal with a mass balance and a momentum balance integrated across the entire flow cross section.

1.2 Schematization of the Cross Section

As its name implies, a one-dimensional mathematical model for flow in open channels contains only one space coordinate, the streamwise coordinate s. As a consequence, the geometric description of the channel cross sections is possible only in terms of bulk parameters applicable to the entire cross section. The variation of the bed elevation and the bed roughness within the cross sections cannot be resolved because that would require a lateral coordinate.

The characterization of the cross section requires a distinction between the *transport* or *conveyance* of water, on the one hand, and its *storage*, on the other. There are situations

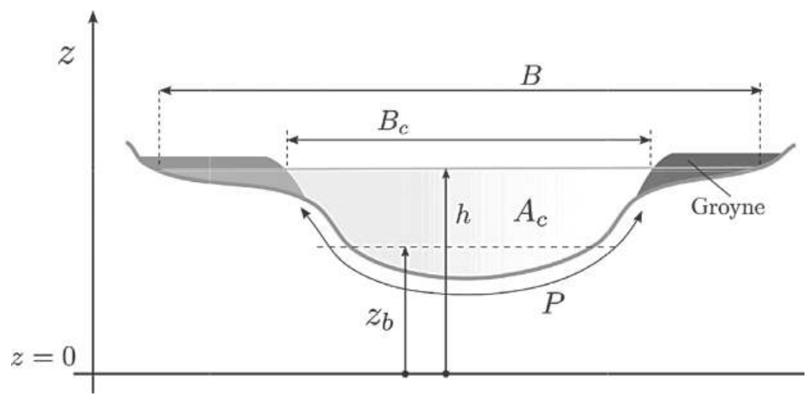

Fig. 1.2 Open conduit: cross section with groynes

where only a part of the wetted cross section contributes significantly to the conveyance. A typical example is provided by a river with a sequence of groynes normal to the flow, where the spaces between the groynes do contribute to the storage capacity but – in the case of low or moderate water levels – not to the conveyance capacity. In those cases it is necessary to distinguish between these two functions.

We designate the area, the width of the free surface and the mean depth of the conveyance cross section as A_c, B_c and d, respectively, where $d = A_c/B_c$. The wetted perimeter is P. See Figure 1.2. Storage takes place through a rise of the free surface, also between the groynes, so that in this respect the total width of the free surface (B) is the relevant parameter. It will be clearly indicated where we use the distinction between the total cross section and that of the conveyance part.

The bed elevation above the horizontal reference plane $z = 0$, averaged over the conveyance cross section, is denoted as z_b. The elevation of the free surface above the reference plane is designated as h. As stated above, and justified in Section 1.4.1, it is assumed to be laterally uniform at all times.

We use a length coordinate s along the streamwise axis that may be weakly curved and gently sloping. The longitudinal slope of the bed ($\tan\beta$), if nonzero, is assumed to be very small, allowing the approximations $\tan\beta \approx \beta$, $\sin\beta \approx \beta$ and $\cos\beta \approx 1$.

1.3 Mass Balance

Consider the mass in a control volume consisting of a slice of a water course with length Δs, containing the entire wet area of the cross section, from bed to free surface. See Figure 1.3.

Because of the free surface, pressure variations in environmental water systems are very limited. Therefore, we can neglect pressure-induced density variations. The water can then be considered as *incompressible*. In that case, the mass balance reduces to a *volume*

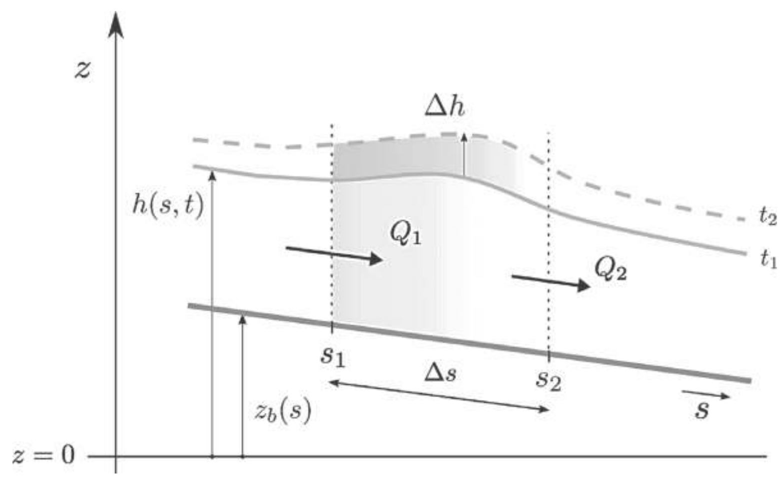

Fig. 1.3 Open conduit: longitudinal transect

balance, also called the *continuity equation*. To derive it, we consider the change in volume of the water in the control volume in a short time interval from $t = t_1$ to $t = t_2 = t_1 + \Delta t$.

The *volume flux* or *discharge* Q in a channel is defined as the volume of water passing a given cross section in a unit of time:

$$Q = \int\int u_s \, \mathrm{d}A = UA_c \tag{1.1}$$

in which u_s is the streamwise velocity at a point and U its value averaged over the conveyance cross section. The net influx of volume of water into the control volume, in the considered short time interval with duration Δt, is

$$(Q_1 - Q_2)\ \Delta t = -\Delta Q \, \Delta t \tag{1.2}$$

Suppose this is positive, i.e. there is more inflow than outflow. The difference is stored in the control volume, giving rise to an increase of the stored volume equal to $\Delta V = \Delta A \Delta s$ (see Figure 1.3).

Equating this storage to the net inflow yields $\Delta A \, \Delta s = -\Delta Q \, \Delta t$. Dividing by Δt and Δs, and taking the limit for $\Delta t \to 0$ and $\Delta s \to 0$, yields

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0 \tag{1.3}$$

The storage is effected through a rise of the free surface by an amount Δh. Using the total width B of the free surface (not only that of the conveyance area) gives $\Delta A = B\Delta h$ (Figure 1.3), with which Eq. (1.3) can be written as

$$B\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial s} = 0 \tag{1.4}$$

For given geometry of the cross section, which may vary with the downstream location s, the free-surface width B varies with time in a known manner through the time variation of h: $B = B(s, h(s,t))$. Therefore, Eq. (1.4), expressing mass conservation for the water (considered incompressible), is our first equation linking variations of the two unknowns Q and h. The second one, to be derived below, expresses momentum conservation.

The continuity equation (1.4) has been derived on the basis of a storage capacity that is continuously distributed along the length of the water course. However, it is not uncommon that there are lateral basins in communication with the main channel, such as harbours, remnants of previous meanders of a river, or dredged sand pits. These provide a discrete storage capacity that has to be taken into account in the overall mass balance. Some such basins can be seen in Figure 1.1.

Assuming that these basins are small compared with the length over which the exterior water level varies, the free surface inside can be assumed to be horizontal at all times, so that the surface elevation is a function of time only, written as $h_b(t)$. Moreover, if the entrance is sufficiently short and wide to allow an unobstructed in- and outflow, the water level in the basin will equal the exterior water level. Chapter 6 elaborates on these approximations in detail in the context of tidal basins.

The storage that can take place in a side basin can be incorporated in the continuity equation (1.4) by a local enlargement of the storage width B over a short longitudinal interval of the channel with length Δs, containing the connection with the basin, as sketched in Figure 1.4. In order to obtain a correct representation of the total storage through the continuity equation on this interval, B must locally be increased with $A_b/\Delta s$.

1.4 Equations of Motion

The formulation of Newton's second law for a flowing mass of water leads to a so-called *equation of motion*, which expresses a *balance between inertia, forcing and resistance*, where each of these in turn can consist of a number of contributions. It is important to be aware of this and to check the meaning of the various terms when writing or reading

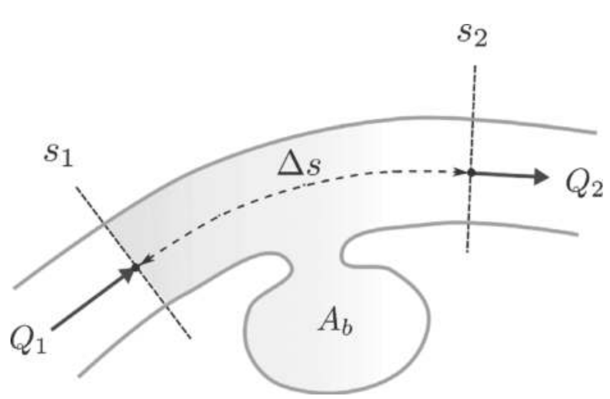

Fig. 1.4 Storage basin connected to a river reach

an equation of motion. It is also important for a good understanding of the phenomena involved to check whether one or more terms is negligible compared with another (in an equation consisting of three or more terms). We discuss this extensively in Chapter 2.

We will initially ignore flow resistance and start from the Euler equations for the acceleration of a fluid particle of an ideal (inviscid) fluid of constant density (ρ) under the action of gravity with gravitational acceleration g. The derivation of the Euler equations can be found in elementary textbooks on fluid mechanics and is not repeated here.

1.4.1 Euler Equations

We present the Euler equations in so-called natural coordinates, defined as follows. The streamwise coordinate is s; the normal coordinate is n, which lies in the local plane of curvature of the flow, pointing to the center of curvature; and the bi-normal coordinate is b, which is perpendicular to the plane of the local curvature (the so-called osculation plane); see Figure 1.5. For the nearly horizontal flows considered here, as in rivers and tidal channels, the bi-normal is nearly vertical; in fact, we will use the vertical coordinate z as the bi-normal coordinate.

Designating the respective particle velocity components as u_s, u_n and u_z, and the radius of curvature as r, and treating g and ρ as constants, the Euler equations can be written as

$$\frac{\mathrm{D}u_s}{\mathrm{D}t} = \frac{\partial u_s}{\partial t} + u_s \frac{\partial u_s}{\partial s} = -g\,\frac{\partial (z + p/\rho g)}{\partial s} \tag{1.5}$$

$$\frac{\mathrm{D}u_n}{\mathrm{D}t} = \frac{\partial u_n}{\partial t} + \frac{u_s^2}{r} = -g\,\frac{\partial (z + p/\rho g)}{\partial n} \tag{1.6}$$

$$\frac{\mathrm{D}u_z}{\mathrm{D}t} = \frac{\partial u_z}{\partial t} = -g\,\frac{\partial (z + p/\rho g)}{\partial z} \tag{1.7}$$

The total derivatives $\mathrm{D}u_s/\mathrm{D}t$ etc. signify the acceleration of a fluid particle following its motion, and p is the fluid pressure. The right-hand sides represent the forcing per unit mass due to gravity and pressure gradients, often expressed as the gradient of the so-called

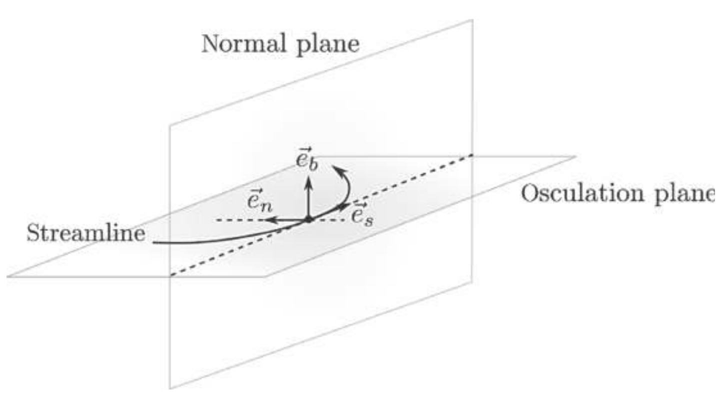

Fig. 1.5 Natural coordinate system

Cambridge University Press
978-1-107-15029-4 — Unsteady Flow in Open Channels
Jurjen A. Battjes , Robert Jan Labeur
Excerpt
More Information

piezometric head h_p, defined as $h_p \equiv z + p/\rho g$. The Euler equations in three dimensions form the basis for the computation of rapidly varying flows.

The assumption of long waves implies that the vertical accelerations are neglected. It then follows from Eq. (1.7) that in this approximation the piezometric head is considered to be uniform in the vertical, or that *the vertical pressure distribution is hydrostatic*. The validity of this approximation is investigated quantitatively in Section 2.2.

The value of h_p at the free surface, and therefore at all points in the vertical, is $(h + p_{\text{atm}}/\rho g)$, in which h is the height of the free surface above the reference level $z = 0$, and p_{atm} is the atmospheric pressure at the air–water interface. Using this, and neglecting horizontal variations of the atmospheric pressure for the time being, the Euler equations in the longitudinal and lateral directions can be written as

$$\frac{\partial u_s}{\partial t} + u_s \frac{\partial u_s}{\partial s} = -g \frac{\partial h}{\partial s} \tag{1.8}$$

$$\frac{\partial u_n}{\partial t} + \frac{u_s^2}{r} = -g \frac{\partial h}{\partial n} \tag{1.9}$$

Note that the combined forcing by gravity and pressure gradients is now expressed in terms of the slope of the free surface, and is constant over the vertical. This is illustrated in Figure 1.6, showing a slice of water and the pressure forces acting on both of its sides. At a given elevation, the slope of the water surface gives rise to different pressures at both sides of the slice, but in the case of hydrostatic pressure, the difference δp is constant over the vertical.

Regarding Eq. (1.9), for the lateral motion, it is relevant to note that throughout this book we consider flows in relatively narrow conduits, so that the flow direction at each point is constant, from which it follows that $\partial u_n/\partial t = 0$. Bends force the flow to change direction. The associated centripetal acceleration u_s^2/r is forced by a lateral slope of the free surface, the latter being higher at the outside of the bend and lower at the inside. These lateral variations in the height of the free surface are crucial in detailed considerations of the spiral flow in bends, but their net effect on the large-scale streamwise motion, with which we are

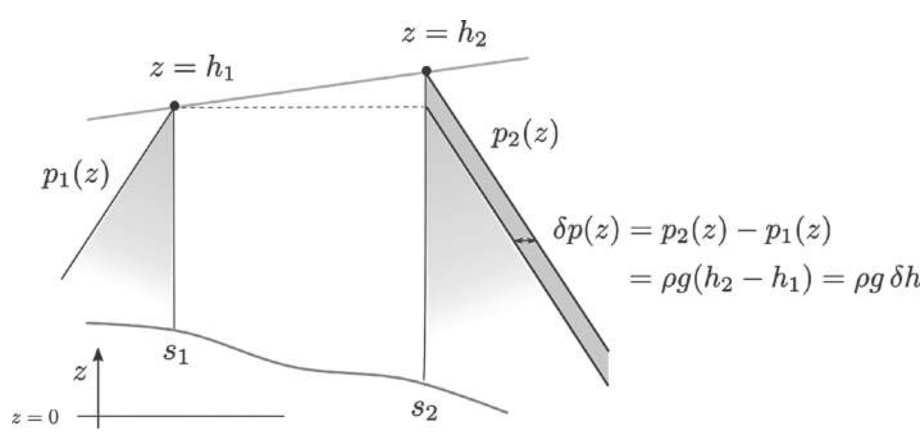

Fig. 1.6 Hydrostatic pressure and net horizontal forcing

concerned, is ignored. This implies that the forcing is considered uniform in the entire cross section and that only streamwise variations in surface elevation are taken into account. Therefore, in the remainder we ignore Eq. (1.9) and continue with Eq. (1.8), considering h to vary with t and s only.

So far we have considered the particle acceleration at a point of the cross section. In order to arrive at a one-dimensional model, we need cross-sectionally integrated or averaged values, as we did for the mass balance. The forcing is uniform in the cross section and, still neglecting resistance and effects of flow curvature in bends, so are the local particle accelerations and velocities. Therefore, in this approximation, Eq. (1.8) also applies to the cross-sectionally averaged flow velocity U, simply by replacing u_s with U:

$$\frac{\mathrm{D}U}{\mathrm{D}t} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial s} = -g\,\frac{\partial h}{\partial s} \tag{1.10}$$

1.4.2 Flow Resistance

Our next task is to introduce flow resistance, which was left out of consideration in the above. In the context of a one-dimensional model, we consider boundary resistance to the bulk of the flow through the (conveyance) cross section, rather than internal flow resistance.

Let ΔW be the boundary resistance experienced by the water in the slice considered in Figure 1.3, with length Δs. Its value per unit area, averaged over the wetted boundary of the conveyance cross section, has the nature of a shear stress, written as τ_b (although, as we will see, processes other than local shear stress can contribute to the resistance): $\tau_b = \Delta W/(P\Delta s)$, in which P is the length of the wetted perimeter of the conveyance cross section.

The resistance per unit mass, $\Delta W/\,(\rho A_c \Delta s)$, can be written as $\tau_b/\rho R$, in which $R = A_c/P$, the so-called *hydraulic radius* of the conveyance cross section. For shallow, wide cross sections, the hydraulic radius is approximately equal to the laterally averaged depth of flow: $R \approx d$.

Adding the resistance force per unit mass (i.e. $-\tau_b/\rho R$) to the right-hand side of Eq. (1.10) yields the following *balance between inertia, forcing and resistance*:

$$\frac{\mathrm{D}U}{\mathrm{D}t} + g\frac{\partial h}{\partial s} + \frac{\tau_b}{\rho R} = 0 \tag{1.11}$$

For future reference, we write this equation in an alternative form as

$$\frac{\mathrm{D}U}{\mathrm{D}t} = g(i_s - i_f) \tag{1.12}$$

in which i_s is the free-surface slope, defined as

$$i_s \equiv -\frac{\partial h}{\partial s} \tag{1.13}$$

and i_f is the so-called friction slope, defined as

$$i_f \equiv \frac{\tau_b}{\rho g R} \tag{1.14}$$

Eq. (1.12) captures the flow dynamics in a nutshell, showing at a glance that the fluid acceleration results from an imbalance between the driving force and the resistance. For steady, uniform flow, these are in balance.

Reverting to Eq. (1.11), with the expanded form of the acceleration, we obtain

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial s} + g\frac{\partial h}{\partial s} + \frac{\tau_b}{\rho R} = 0 \tag{1.15}$$

This equation seems to have been first derived by De Saint-Venant (1871), although his name is usually associated with a similar equation in terms of the discharge Q; see Eq. (1.19) below.

In order to complete the formulation, we need a constitutive relationship between the resistance and the flow velocity. It is usually assumed that the resistance varies with the average velocity U as in uniform, steady turbulent flows. The flows of interest have high Reynolds numbers, typically of the order of 10^6, so that viscous effects can be ignored in the modelling, and the resistance varies in proportion to the square of the flow velocity, as in

$$\tau_b = c_f \rho |U| U \tag{1.16}$$

in which c_f is a dimensionless resistance coefficient. Methods of its estimation are dealt with in Section 9.3, on uniform flow. Typical values are in the range of 0.002–0.006. We will often use 0.004 in numerical examples.

The friction slope corresponding to Eq. (1.16) is

$$i_f = c_f \frac{|U|U}{gR} \tag{1.17}$$

Substituting Eq. (1.16) in to Eq. (1.15) yields

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial s} + g\frac{\partial h}{\partial s} + c_f\frac{|U|U}{R} = 0 \tag{1.18}$$

which completes the derivation of the *acceleration equation* for the averaged flow velocity U including boundary resistance.

In the derivation of Eq. (1.18) it has been assumed that the resistance is evenly distributed over the wetted perimeter of the cross section. This is a good approximation if the variations of depth, bed roughness and flow velocities within a cross section are moderate, but less so if they vary significantly.

In periods of high water in rivers, for example, the flood plains are covered and contribute to storage as well as conveyance, but the depth of flow over the flood plains is much less than it is in the main channel, and the resistance is usually much greater (due to vegetation, buildings etc.). Similar situations occur in estuaries with their main channels and shoals and in tidal channels bordered by tidal flats. These lateral variations in a cross section can be accounted for in the schematization by dividing the cross section into two or more subsections, each of them with its own characteristic width, depth and roughness. (An application of this for tidal propagation is presented in Section 7.4.2.)

1.4.3 Momentum Balance

Since the discharge Q is one of our two primary variables (h being the other one), we will now derive a momentum balance in terms of Q. For algebraic simplicity we will temporarily make no distinction between the total cross section and the conveyance part.

We could derive the momentum balance from Eq. (1.18) by substituting $U = Q/A_c$, but that requires several intermediate steps. It is simpler to start afresh by considering cross-sectionally integrated quantities from the outset, which at the same time allows us to express the effect of variations of the local particle velocity in a cross section. To this end, we consider (again) a control volume covering the entire wet cross section, as in Figure 1.3.

The streamwise momentum per unit length is the cross-sectional integral of ρu_s, which equals ρQ. The streamwise advection of streamwise momentum through a cross section is the cross-sectional integral of $\rho(u_s)^2$, which is written as $\alpha\rho U^2 A_c$, or $\alpha\rho Q^2/A_c$, in which α is a coefficient expressing the effect of the variations of the particle velocity within the cross section, defined as the cross-sectional average of $(u_s/U)^2$. Its value is always more than unity, but the deviations are usually rather small, and α is mostly ignored in practice. The driving force and the boundary resistance, both per unit mass, are given by the last two terms of Eq. (1.18).

Collecting the preceding contributions, we finally obtain the balance of streamwise momentum as follows (dropping the constant mass density ρ):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s}\left(\frac{\alpha Q^2}{A_c}\right) + gA_c\frac{\partial h}{\partial s} + c_f\frac{|Q|Q}{A_c R} = 0 \tag{1.19}$$

In some applications, it is necessary to account for effects of the atmosphere on the flow. This requires additional terms in the momentum balance, shown in Box 1.1.

In the derivation of Eq. (1.19), it was assumed that the entire cross section contributes to the conveyance, such that $A = A_c$ and $B = B_c$. If this is not the case, a lateral exchange flow occurs between the main channel and the flood plains (in the case of river flow), requiring an additional term in Eq. (1.19), which, if inserted in the left-hand side, is given by $\rho U(B - B_c)(\partial h/\partial t)$; see Box 1.2.

Box 1.1 Inclusion of atmospheric forcings

In some applications, effects of a variable atmospheric pressure and of wind-induced shear at the free surface must be taken into account. These influences can be included in Eq. (1.19) by adding $p_{\text{atm}}/\rho g$ to h and by adding the streamwise component of the wind-induced shear force per unit length at the free surface (to be divided by ρ) given by $\tau_s B_c \cos\psi$, in which B_c is the width of the conveyance cross section, ψ is the angle enclosed between the wind direction and the direction of the flow, and τ_s is the wind-induced shear stress at the free surface: $\tau_s = c_D\,\rho_a\,W_{10}^2$ in which c_D is the wind drag coefficient (of order 0.001–0.002), ρ_a is the air density, and W_{10} is the 10-min averaged windspeed at a height of 10 m above the mean water level.