Commutative Algebra and Noncommutative Algebraic Geometry, I
MSRI Publications
Volume **67**, 2015

When is a squarefree monomial ideal of linear type?

ALI ALILOOEE AND SARA FARIDI

In 1995 Villarreal gave a combinatorial description of the equations of Rees algebras of quadratic squarefree monomial ideals. His description was based on the concept of closed even walks in a graph. In this paper we will generalize his results to all squarefree monomial ideals by defining even walks in a simplicial complex. We show that simplicial complexes with no even walks have facet ideals that are of linear type, generalizing Villarreal's work.

1. Introduction

Rees algebras are of special interest in algebraic geometry and commutative algebra since they describe the blowing up of the spectrum of a ring along the subscheme defined by an ideal. The Rees algebra of an ideal can also be viewed as a quotient of a polynomial ring. If I is an ideal of a ring R, we denote the Rees algebra of I by $R[It]$, and we can represent $R[It]$ as S/J where S is a polynomial ring over R. The ideal J is called the *defining ideal* of $R[It]$. Finding generators of J is difficult and crucial for better understanding $R[It]$. Many authors have worked to gain better insight into these generators in special classes of ideals, such as those with special height, special embedding dimension and so on.

When I is a monomial ideal, using methods from Taylor's thesis [1966] one can describe the generators of J as binomials. Using this fact, Villarreal [1995] gave a combinatorial characterization of J in the case of degree 2 squarefree monomial ideals. His work led Fouli and Lin [2015] to consider the question of characterizing generators of J when I is a squarefree monomial ideal in any degree. With this purpose in mind we define simplicial even walks, and show that for all squarefree monomial ideals, they identify generators of J that may be obstructions to I being of linear type. We show that in dimension 1, simplicial even walks are the same as closed even walks of graphs. We then further investigate properties of simplicial even walks, and reduce the problem

Alilooee was supported by a Killam Scholarship and Faridi's research was search supported by NSERC.

of checking whether an ideal is of linear type to identifying simplicial even walks. At the end of the paper we give a new proof for Villarreal's theorem (Corollary 4.10).

2. Rees algebras and their equations

Let I be a monomial ideal in a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$. We denote the *Rees algebra* of $I = (f_1, \dots, f_q)$ by $R[It] = R[f_1 t, \dots, f_q t]$ and consider the homomorphism ψ of algebras

$$\psi : R[T_1, \dots, T_q] \longrightarrow R[It], \quad T_i \mapsto f_i t.$$

If J is the kernel of ψ, we can consider the Rees algebra $R[It]$ as the quotient of the polynomial ring $R[T_1, \dots, T_q]$. The ideal J is called the *defining ideal* of $R[It]$ and its minimal generators are called the *Rees equations* of I. These equations carry a lot of information about $R[It]$; see for example [Vasconcelos 1994] for more details.

Definition 2.1. For integers $s, q \geq 1$ we define

$$\mathcal{I}_s = \{(i_1, \dots, i_s) : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_s \leq q\} \subset \mathbb{N}^s.$$

Let $\alpha = (i_1, \dots, i_s) \in \mathcal{I}_s$ and $f_1, \dots, f_q$ be monomials in R and $T_1, \dots, T_q$ be variables. We use the following notation throughout, where $t \in \{1, \dots, s\}$.

- $\mathrm{Supp}(\alpha) = \{i_1, \dots, i_s\}$.
- $\hat{\alpha}_{i_t} = (i_1, \dots, \hat{i}_t, \dots, i_s)$.
- $T_\alpha = T_{i_1} \dots T_{i_s}$ and $\mathrm{Supp}(T_\alpha) = \{T_{i_1}, \dots, T_{i_s}\}$.
- $f_\alpha = f_{i_1} \dots f_{i_s}$.
- $\hat{f}_{\alpha_t} = f_{i_1} \dots \hat{f}_{i_t} \dots f_{i_s} = f_\alpha / f_{i_t}$.
- $\widehat{T}_{\alpha_t} = T_{i_1} \dots \widehat{T}_{i_t} \dots T_{i_s} = T_\alpha / T_{i_t}$.
- $\alpha_t(j) = (i_1, \dots, i_{t-1}, j, i_{t+1}, \dots, i_s)$, for $j \in \{1, 2, \dots, q\}$ and $s \geq 2$.

For an ideal $I = (f_1, \dots, f_q)$ of R the defining ideal J of $R[It]$ is graded and

$$J = J_1' \oplus J_2' \oplus \cdots,$$

where J_s' for $s \geq 1$ is the R-module.

The ideal I is said to be *of linear type* if $J = (J_1')$; in other words, the defining ideal of $R[It]$ is generated by linear forms in the variables $T_1, \dots, T_q$.

Definition 2.2. Let $I = (f_1, \dots, f_q)$ be a monomial ideal, $s \geq 2$ and $\alpha, \beta \in \mathcal{I}_s$. We define

$$T_{\alpha,\beta}(I) = \left(\frac{\mathrm{lcm}(f_\alpha, f_\beta)}{f_\alpha}\right) T_\alpha - \left(\frac{\mathrm{lcm}(f_\alpha, f_\beta)}{f_\beta}\right) T_\beta. \tag{2-1}$$

When I is clear from the context we use $T_{\alpha,\beta}$ to denote $T_{\alpha,\beta}(I)$.

Proposition 2.3 [Taylor 1966]. *Let $I = (f_1, \dots, f_q)$ be a monomial ideal in R and J be the defining ideal of $R[It]$. Then for $s \geq 2$ we have*

$$J'_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s \rangle.$$

Moreover, if $m = \gcd(f_1, \dots, f_q)$ and $I' = (f_1/m, \dots, f_q/m)$, then for every $\alpha, \beta \in \mathcal{I}_s$ we have

$$T_{\alpha,\beta}(I) = T_{\alpha,\beta}(I'),$$

and hence $R[It] = R[I't]$.

In light of Proposition 2.3, we will always assume that if $I = (f_1, \dots, f_q)$ then

$$\gcd(f_1, \dots, f_q) = 1.$$

We will also assume $\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing$, since otherwise $T_{\alpha,\beta}$ reduces to those with this property. This is because if $t \in \mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta)$ then $T_{\alpha,\beta} = T_t T_{\hat{\alpha}_t,\hat{\beta}_t}$.

For this reason we define

$$J_s = \langle T_{\alpha,\beta}(I) : \alpha, \beta \in \mathcal{I}_s, \mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing \rangle \tag{2-2}$$

as an R-module. Clearly $J = J_1 S + J_2 S + \cdots$.

Definition 2.4. Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in R and J be the defining ideal of $R[IT]$, $s \geq 2$, and $\alpha = (i_1, \dots, i_s)$, $\beta = (j_1, \dots, j_s) \in \mathcal{I}_s$. We call $T_{\alpha,\beta}$ *redundant* if it is a redundant generator of J, coming from lower degree; i.e.,

$$T_{\alpha,\beta} \in J_1 S + \cdots + J_{s-1} S.$$

3. Simplicial even walks

By using the concept of closed even walks in graph theory, Villarreal [1995] classified all Rees equations of edge ideals of graphs in terms of closed even walks. In this section our goal is to define an even walk in a simplicial complex in order to classify all irredundant Rees equations of squarefree monomial ideals. Motivated by the work of S. Petrović and D. Stasi [2014], we generalize closed even walks from graphs to simplicial complexes.

We begin with basic definitions that we will need later.

Definition 3.1. A *simplicial complex* on vertex set $V = \{x_1, \dots, x_n\}$ is a collection Δ of subsets of V satisfying

(1) $\{x_i\} \in \Delta$ for all i,

(2) $F \in \Delta, G \subseteq F \Longrightarrow G \in \Delta$.

The set V is called the *vertex set* of Δ and we denote it by $V(\Delta)$. The elements of Δ are called *faces* of Δ and the maximal faces under inclusion are called *facets*. We denote the simplicial complex Δ with facets $F_1, \dots, F_s$ by $\langle F_1, \dots, F_s \rangle$. We denote the set of facets of Δ with $\text{Facets}(\Delta)$. A *subcollection* of a simplicial complex Δ is a simplicial complex whose facet set is a subset of the facet set of Δ.

Definition 3.2. Let Δ be a simplicial complex with at least three facets, ordered as $F_1, \dots, F_q$. Suppose $\bigcap F_i = \varnothing$. With respect to this order Δ is

(i) an *extended trail* if

$$F_i \cap F_{i+1} \neq \varnothing \quad i = 1, \dots, q \mod q;$$

(ii) a *special cycle* [Herzog et al. 2008] if Δ is an extended trail in which

$$F_i \cap F_{i+1} \not\subset \bigcup_{j \notin \{i, i+1\}} F_j \quad i = 1, \dots, q \mod q;$$

(iii) a *simplicial cycle* [Caboara et al. 2007] if Δ is an extended trail in which

$$F_i \cap F_j \neq \varnothing \Leftrightarrow j \in \{i+1, i-1\} \quad i = 1, \dots, q \mod q.$$

We say that Δ is an extended trail (or special or simplicial cycle) if there is an order on the facets of Δ such that the specified conditions hold on that order. Note that

$$\{\text{simplicial cycles}\} \subseteq \{\text{special cycles}\} \subseteq \{\text{extended trails}\}.$$

Definition 3.3 (simplicial trees and simplicial forests [Caboara et al. 2007; Faridi 2002]). A simplicial complex Δ is called a *simplicial forest* if Δ contains no simplicial cycle. If Δ is also connected, it is called a *simplicial tree*.

Definition 3.4 [Zheng 2004, Lemma 3.10]. Let Δ be a simplicial complex. The facet F of Δ is called a *good leaf* of Δ if the set $\{H \cap F;\ H \in \text{Facets}(\Delta)\}$ is totally ordered by inclusion.

Good leaves were first introduced by X. Zheng in her PhD thesis [2004] and later in [Caboara et al. 2007]. The existence of a good leaf in every tree was proved by J. Herzog, T. Hibi, N. V. Trung and X. Zheng:

Theorem 3.5 [Herzog et al. 2008, Corollary 3.4]. *Every simplicial forest contains a good leaf.*

Let $I = (f_1, \dots, f_q)$ be a squarefree monomial ideal in $R = \mathbb{K}[x_1, \dots, x_n]$. The *facet complex* $\mathcal{F}(I)$ associated to I is a simplicial complex with facets $F_1, \dots, F_s$, where for each i,

$$F_i = \{x_j : x_j \mid f_i,\ 1 \leq j \leq n\}.$$

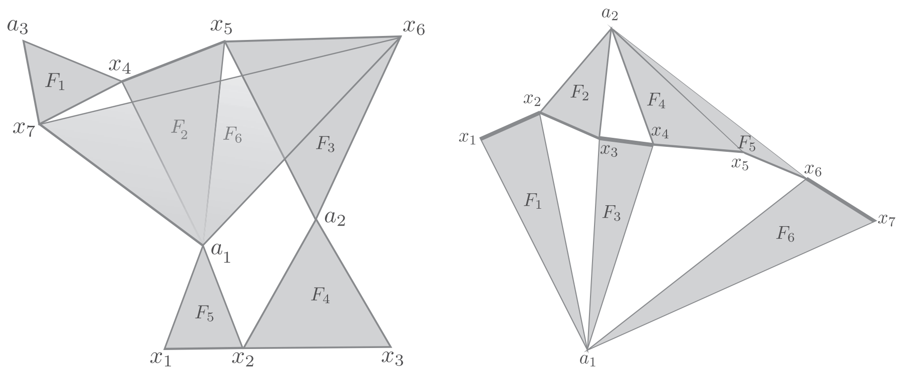


Figure 1. Left: an even walk. Right: not an even walk.

The *facet ideal* of a simplicial complex Δ is the ideal generated by the products of the variables labeling the vertices of each facet of Δ; in other words

$$\mathcal{F}(\Delta) = \left(\prod_{x \in F} x : \ F \text{ is a facet of } \Delta \right).$$

Definition 3.6 (degree). Let $\Delta = \langle F_1, \dots, F_q \rangle$ be a simplicial complex, $\mathcal{F}(\Delta) = (f_1, \dots, f_q)$ be its facet ideal and $\alpha = (i_1, \dots, i_s) \in \mathcal{I}_s$, $s \geq 1$. We define the *α-degree for a vertex x* of Δ to be

$$\deg_\alpha(x) = \max\{m : x^m \mid f_\alpha\}.$$

Example 3.7. Consider Figure 1 (left), where

$$F_1 = \{x_4, x_7, a_3\}, \quad F_2 = \{x_4, x_5, a_1\}, \quad F_3 = \{x_5, x_6, a_2\},$$
$$F_4 = \{x_2, x_3, a_2\}, \quad F_5 = \{x_1, x_2, a_1\}, \quad F_6 = \{x_6, x_7, a_1\}.$$

If we consider $\alpha = (1, 3, 5)$ and $\beta = (2, 4, 6)$ then $\deg_\alpha(a_1) = 1$ and $\deg_\beta(a_1) = 2$.

Suppose $I = (f_1, \dots, f_q)$ is a squarefree monomial ideal in R with $\Delta = \langle F_1, \dots, F_q \rangle$ its facet complex and let $\alpha, \beta \in \mathcal{I}_s$ where $s \geq 2$ is an integer. We set $\alpha = (i_1, \dots, i_s)$ and $\beta = (j_1, \dots, j_s)$ and consider the following sequence of not necessarily distinct facets of Δ:

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \dots, F_{i_s}, F_{j_s}.$$

Then (2-1) becomes

$$T_{\alpha,\beta}(I) = \left(\prod_{\deg_\alpha(x) < \deg_\beta(x)} x^{\deg_\beta(x) - \deg_\alpha(x)} \right) T_\alpha - \left(\prod_{\deg_\alpha(x) > \deg_\beta(x)} x^{\deg_\alpha(x) - \deg_\beta(x)} \right) T_\beta, \quad (3\text{-}1)$$

where the products vary over the vertices x of $\mathcal{C}_{\alpha,\beta}$.

Definition 3.8 (simplicial even walk). Let $\Delta = \langle F_1, \dots, F_q \rangle$ be a simplicial complex and let $\alpha = (i_1, \dots, i_s)$, $\beta = (j_1, \dots, j_s) \in \mathcal{I}_s$, where $s \geq 2$. The following sequence of not necessarily distinct facets of Δ

$$\mathcal{C}_{\alpha,\beta} = F_{i_1}, F_{j_1}, \dots, F_{i_s}, F_{j_s}$$

is called a *simplicial even walk*, or simply "even walk", if for every $i \in \mathrm{Supp}(\alpha)$ and $j \in \mathrm{Supp}(\beta)$ we have

$$F_i \backslash F_j \not\subset \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) > \deg_\beta(x)\},$$
$$F_j \backslash F_i \not\subset \{x \in \mathrm{V}(\Delta) : \deg_\alpha(x) < \deg_\beta(x)\}.$$

If $\mathcal{C}_{\alpha,\beta}$ is connected, we call the even walk $\mathcal{C}_{\alpha,\beta}$ a *connected* even walk.

Remark 3.9. It follows from the definition, if $\mathcal{C}_{\alpha,\beta}$ is an even walk then

$$\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta) = \varnothing.$$

Example 3.10. In Figure 1 by setting $\alpha = (1, 3, 5)$, $\beta = (2, 4, 6)$ we have

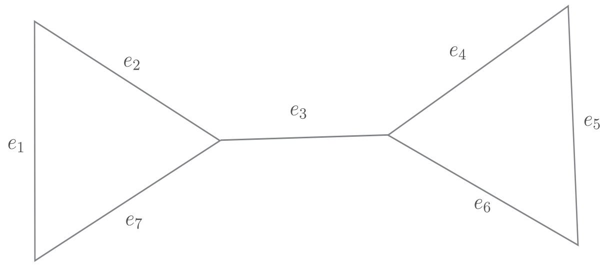
$$F_1 \backslash F_2 = \{x_1, a_1\} = \{x : \deg_\alpha(x) > \deg_\beta(x)\}.$$

Remark 3.11. It turns out that a minimal even walk (that is, one not properly containing another even walk) can have repeated facets. For instance, the bicycle graph in Figure 2 is a minimal even walk, because of Corollary 3.24 below, but it has a pair of repeated edges.

Proposition 3.12 (structure of even walks). *Let* $\mathcal{C}_{\alpha,\beta} = F_1, F_2, \dots, F_{2s}$ *be an even walk.*

(i) *If* $i \in \mathrm{Supp}(\alpha)$ (*or* $i \in \mathrm{Supp}(\beta)$) *there exist distinct* $j, k \in \mathrm{Supp}(\beta)$ (*or* $j, k \in \mathrm{Supp}(\alpha)$) *such that*

$$F_i \cap F_j \neq \varnothing \quad \textit{and} \quad F_i \cap F_k \neq \varnothing. \tag{3-2}$$



$\alpha = (1, 3, 3, 5) \qquad \beta = (2, 4, 6, 7)$

Figure 2. A minimal even walk with repeated facets.

(ii) *The simplicial complex* $\langle \mathcal{C}_{\alpha,\beta} \rangle$ *contains an extended trail of even length labeled* $F_{v_1}, F_{v_2}, \ldots, F_{v_{2l}}$ *where*

$$v_1, \ldots, v_{2l-1} \in \operatorname{Supp}(\alpha) \quad \textit{and} \quad v_2, \ldots, v_{2l} \in \operatorname{Supp}(\beta).$$

Proof. To prove (i) let $i \in \operatorname{Supp}(\alpha)$, and consider the following set

$$\mathcal{A}_i = \{j \in \operatorname{Supp}(\beta) : F_i \cap F_j \neq \varnothing\}.$$

We only need to prove that $|\mathcal{A}_i| \geq 2$.

Suppose $|\mathcal{A}_i| = 0$ then for all $j \in \operatorname{Supp}(\beta)$ we have

$$F_i \setminus F_j = F_i \subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\},$$

because for each $x \in F_i \setminus F_j$ we have $\deg_\beta(x) = 0$ and $\deg_\alpha(x) > 0$; a contradiction.

Suppose $|\mathcal{A}_i| = 1$ so that there is one $j \in \operatorname{Supp}(\beta)$ such that $F_i \cap F_j \neq \varnothing$. So for every $x \in F_i \setminus F_j$ we have $\deg_\beta(x) = 0$. Therefore, we have

$$F_i \setminus F_j \subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\},$$

again a contradiction. So we must have $|\mathcal{A}_i| \geq 2$.

To prove (ii) pick $u_1 \in \operatorname{Supp}(\alpha)$. By using the previous part we can say there are $u_0, u_2 \in \operatorname{Supp}(\beta)$, $u_0 \neq u_2$, such that

$$F_{u_0} \cap F_{u_1} \neq \varnothing \quad \text{and} \quad F_{u_1} \cap F_{u_2} \neq \varnothing.$$

By a similar argument there is $u_3 \in \operatorname{Supp}(\alpha)$ such that $u_1 \neq u_3$ and $F_{u_2} \cap F_{u_3} \neq \varnothing$. We continue this process. Pick $u_4 \in \operatorname{Supp}(\beta)$ such that

$$F_{u_4} \cap F_{u_3} \neq \varnothing \quad \text{and} \quad u_4 \neq u_2.$$

If $u_4 = u_0$, then $F_{u_0}, F_{u_1}, F_{u_2}, F_{u_3}$ is an even length extended trail. If not, we continue this process each time taking

$$F_{u_0}, \ldots, F_{u_n},$$

and picking $u_{n+1} \in \operatorname{Supp}(\alpha)$ (or u_{n+1} $\operatorname{Supp}(\beta)$) if $u_n \in \operatorname{Supp}(\beta)$ (or $u_n \in \operatorname{Supp}(\alpha)$) such that

$$F_{u_{n+1}} \cap F_{u_n} \neq \varnothing \quad \text{and} \quad u_{n+1} \neq u_{n-1}.$$

If $u_{n+1} \in \{u_0, \ldots, u_{n-2}\}$, say $u_{n+1} = u_m$, then the process stops and we have

$$F_{u_m}, F_{u_{m+1}}, \ldots, F_{u_n}$$

is an extended trail. The length of this cycle is even since the indices

$$u_m, u_{m+1}, \ldots, u_n$$

alternately belong to $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ (which are disjoint by our assumption), and if $u_m \in \mathrm{Supp}(\alpha)$, then by construction $u_n \in \mathrm{Supp}(\beta)$ and vice versa. So there are an even length of such indices and we are done.

If $u_{n+1} \notin \{u_0, \dots, u_{n-2}\}$ we add it to the end of the sequence and repeat the same process for $F_{u_0}, F_{u_1}, \dots, F_{u_{n+1}}$. Since $\mathcal{C}_{\alpha,\beta}$ has a finite number of facets, this process has to stop. □

Corollary 3.13. *An even walk has at least* 4 *distinct facets.*

Theorem 3.14. *A simplicial forest contains no simplicial even walk.*

Proof. Assume the forest Δ contains an even walk $\mathcal{C}_{\alpha,\beta}$ where $\alpha, \beta, \in \mathcal{I}_s$ and $s \geq 2$ is an integer. Since Δ is a simplicial forest so is its subcollection $\langle \mathcal{C}_{\alpha,\beta} \rangle$, so by Theorem 3.5 $\langle \mathcal{C}_{\alpha,\beta} \rangle$ contains a good leaf F_0. So we can consider the following order on the facets $F_0, \dots, F_q$ of $\langle \mathcal{C}_{\alpha,\beta} \rangle$:

$$F_q \cap F_0 \subseteq \cdots \subseteq F_2 \cap F_0 \subseteq F_1 \cap F_0. \tag{3-3}$$

Without loss of generality we suppose $0 \in \mathrm{Supp}(\alpha)$. Since $\mathrm{Supp}(\beta) \neq \varnothing$, we can pick $j \in \{1, \dots, q\}$ to be the smallest index with $F_j \in \mathrm{Supp}(\beta)$. Now if $x \in F_0 \setminus F_j$, by (3-3) we will have $\deg_\alpha(x) \geq 1$ and $\deg_\beta(x) = 0$, which shows that

$$F_0 \setminus F_j \subset \{x \in V(\mathcal{C}_{\alpha,\beta}); \deg_\alpha(x) > \deg_\beta(x)\},$$

a contradiction. □

Corollary 3.15. *Every simplicial even walk contains a simplicial cycle.*

An even walk is not necessarily an extended trail. For instance see the following example.

Example 3.16. Let $\alpha = (1, 3, 5, 7)$, $\beta = (2, 4, 6, 8)$ and $\mathcal{C}_{\alpha,\beta} = F_1, \dots, F_8$ as in Figure 3. It can easily be seen that $\mathcal{C}_{\alpha,\beta}$ is an even walk of distinct facets but $\mathcal{C}_{\alpha,\beta}$ is not an extended trail. The main point here is that we do not require that $F_i \cap F_{i+1} \neq \varnothing$ in an even walk which is necessary condition for extended trails. For example $F_4 \cap F_5 \neq \varnothing$ in this case.

On the other hand, every even-length special cycle is an even walk.

Proposition 3.17 (even special cycles are even walks). *If $F_1, \dots, F_{2s}$ is a special cycle* (*under the written order*) *then it is an even walk under the same order.*

Proof. Let $\alpha = (1, 3, \dots, 2s-1)$ and $\beta = (2, 4, \dots, 2s)$, and set $\mathcal{C}_{\alpha,\beta} = F_1, \dots, F_{2s}$. Suppose $\mathcal{C}_{\alpha,\beta}$ is not an even walk, so there is $i \in \mathrm{Supp}(\alpha)$ and $j \in \mathrm{Supp}(\beta)$ such that at least one of the following conditions holds:

$$\begin{aligned} F_i \setminus F_j &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) > \deg_\beta(x)\}, \\ F_j \setminus F_i &\subseteq \{x \in \mathrm{V}(\mathcal{C}_{\alpha,\beta}) : \deg_\alpha(x) < \deg_\beta(x)\}. \end{aligned} \tag{3-4}$$

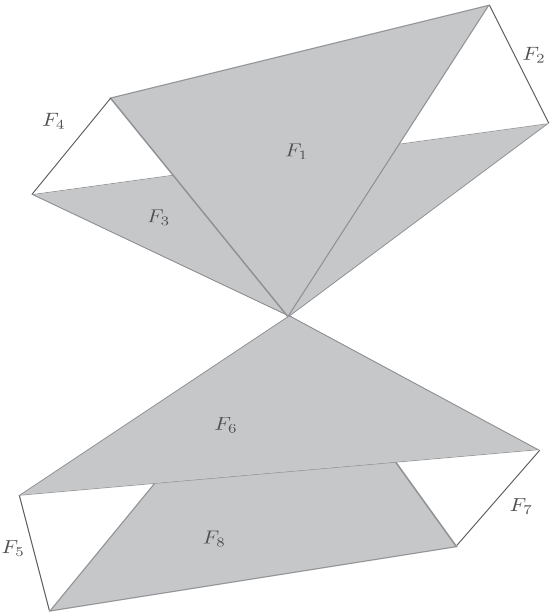


Figure 3. An even walk which is not an extended trail.

Without loss of generality we can assume that the first condition holds. Pick $h \in \{i-1, i+1\}$ such that $h \neq j$. Then by definition of special cycle there is a vertex $z \in F_i \cap F_h$ and $z \notin F_l$ for $l \notin \{i, h\}$. In particular, $z \in F_i \setminus F_j$, but $\deg_\alpha(z) = \deg_\beta(z) = 1$, which contradicts (3-4). □

The converse of Proposition 3.17 is not true: not every even walk is a special cycle, see, for example, Figure 1 (left) or Figure 3, which are not even extended trails. But one can show that it is true for even walks with four facets (see [Alilooee 2014]).

3A. ***The case of graphs.*** We demonstrate that Definition 3.8 in dimension 1 restricts to closed even walks in graph theory. For more details on the graph theory mentioned in this section we refer the reader to [West 1996].

Definition 3.18. Let $G = (\mathrm{V}, E)$ be a graph (not necessarily simple) where V is a nonempty set of vertices and E is a set of edges. A *walk* of length n in G is a list $e_1, e_2, \dots, e_n$ of not necessarily distinct edges such that

$$e_i = \{x_i, x_{i+1}\} \in E \quad \text{for each } i \in \{1, \dots, n-1\}.$$

A walk is called *closed* if its endpoints are the same, i.e., $x_1 = x_n$. The length of a walk $\mathcal{W}$ is denoted by $\ell(\mathcal{W})$. A walk with no repeated edges is called a *trail* and a walk with no repeated vertices or edges is called a *path*. A closed

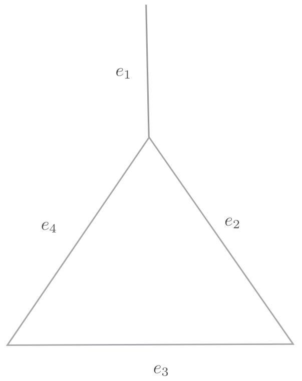


Figure 4. An extended trail that is neither a trail nor a cycle.

walk with no repeated vertices or edges allowed, other than the repetition of the starting and ending vertex, is called a *cycle*.

Lemma 3.19 [West 1996, Lemma 1.2.15 and Remark 1.2.16]. *Let G be a simple graph. Then we have*:

- *Every closed odd walk contains a cycle.*
- *Every closed even walk which has at least one nonrepeated edge contains a cycle.*

Note that in the graph case the special and simplicial cycles are the ordinary cycles. But extended trails in our definition are not necessarily cycles in the case of graphs or even a trail. For instance the graph in Figure 4 is an extended trail, which is not neither a cycle nor a trail, but contains one cycle. This is the case in general.

Theorem 3.20 (Euler's theorem [West 1996]). *If G is a connected graph, then G is a closed walk with no repeated edges if and only if the degree of every vertex of G is even.*

Lemma 3.21. *Let G be a simple graph and let* $\mathcal{C} = e_{i_1}, \dots, e_{i_{2s}}$ *be a sequence of not necessarily distinct edges of G where* $s \geq 2$ *and* $e_i = \{x_i, x_{i+1}\}$ *and* $f_i = x_i x_{i+1}$ *for* $1 \leq i \leq 2s$. *Let* $\alpha = (i_1, i_3, \dots, i_{2s-1})$ *and* $\beta = (i_2, i_4, \dots, i_{2s})$. *Then* $\mathcal{C}$ *is a closed even walk if and only if* $f_\alpha = f_\beta$.

Proof. (⇒) This direction is clear from the definition of closed even walks.

(⇐) We can give to each repeated edge in $\mathcal{C}$ a new label and consider $\mathcal{C}$ as a multigraph (a graph with multiple edges). The condition $f_\alpha = f_\beta$ implies that every $x \in \mathrm{V}(\mathcal{C})$ has even degree, as a vertex of the multigraph $\mathcal{C}$ (a graph containing edges that are incident to the same two vertices). Theorem 3.20 implies that $\mathcal{C}$ is a closed even walk with no repeated edges. Now we revert back