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Riemann Surfaces and Algebraic Curves

A First Course in Hurwitz Theory

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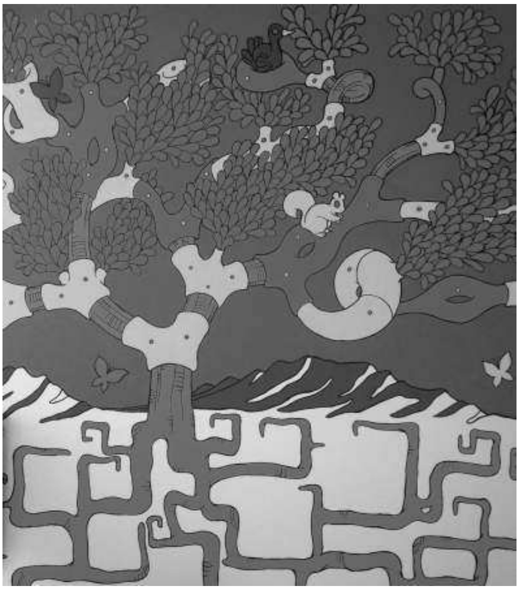

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Introduction

Hurwitz theory is a beautiful algebro-geometric theory that studies maps of Riemann Surfaces. Despite being (relatively) unsophisticated, it is typically unapproachable at the undergraduate level because it ties together several branches of mathematics that are commonly treated separately. This book intends to present Hurwitz theory to an undergraduate audience, paying special attention to the connections between algebra, geometry and complex analysis that it brings about. We illustrate this point by giving an overview of the material in the book.

Hurwitz theory is the enumerative study of analytic functions between Riemann Surfaces – complex compact manifolds of dimension one. A Hurwitz number counts the number of such functions when the appropriate set of discrete invariants is fixed. This has its origin in the 1800s in the work of Riemann, who first had the insight that multi-valued inverses of complex analytic functions can be naturally seen as functions defined on a domain which is locally, but not globally, identifiable with the complex plane: i.e. a Riemann Surface.

Studying analytic functions defined on Riemann Surfaces leads to the geometry of oriented topological surfaces, which Riemann Surfaces are. The local behavior of functions reveals a high degree of structure: analytic functions are ramified coverings; that is, coverings except at a discrete set of points where a phenomenon called *ramification* occurs.

Ramified coverings naturally give rise to monodromy representations, which are homomorphisms from the fundamental group of the punctured target surface to a symmetric group. The ramification at the preimages of a point b in the base is captured by the cycle type of the permutation associated with a small loop winding around the point b.

The count of all such representations can be identified with a coefficient of a specific product of vectors in the class algebra of the symmetric group: with a vector space which has a basis indexed by conjugacy classes. Elements of this basis are given by formal sums of all permutations in the same conjugacy class. A commutative multiplication is then defined by extending the group operation of the symmetric group by bilinearity.

The class algebra is known to be semisimple: it admits a basis with respect to which multiplication is idempotent. Computing the product above in the semisimple basis yields closed formulas for Hurwitz numbers in terms of characters of the symmetric group.

To summarize, the count of analytic functions was translated to a geometric count of topological covers, then to an algebraic count of group homomorphisms, and finally reduced to a representation theoretic computation.

In a different direction, Riemann Surfaces can be degenerated to nodal surfaces by shrinking loops. These nodal surfaces look like "smaller" Riemann Surfaces glued at points, and so degeneration creates infinite families of recursive relations among Hurwitz numbers. We conclude the book by showing that when Hurwitz numbers are encoded as coefficients of a formal power series (a generating function called the *Hurwitz potential*), some of these recursions translate into partial differential equations that are solved by the Hurwitz potential.

Whether this summary makes perfect sense or no sense at all depends on the background of the reader. In any case, we hope that at least two things are apparent: first, that keywords from several different undergraduate courses have been used; and second, that no exceptionally sophisticated term appeared.

This book arises from two experimental undergraduate courses that the first author taught at Colorado State University in 2014 and 2015. The courses were offered as a follow-up to classes in topology and differential geometry; a main goal was to depart from the structure of a traditional course and offer the students a mode of approaching the study of mathematics closer to that of a researcher facing a new problem.

At a school like Colorado State University, most advanced math majors have typically taken semester-long courses in some of the areas mentioned in the above synopsis, and typically have not taken all those courses. There is some analogy with the situation that mathematical researchers are in when they tackle an open problem. First of all, translation and reformulation of a problem is often a very important tool in mathematical research. Problems that are too difficult when studied in a certain way may become approachable when the point of view is changed. When mathematical researchers translate a question in order to find ways to solve it, they are often taken into mathematical areas out of their comfort zone. And they don't have the opportunity to take a semester-long course, or to read a whole book on each topic that they use, but must be able efficiently to develop a working understanding of the aspects needed for their problem.

This analogy informed the way we structured the narration of our story. We have background chapters that introduce complex analysis, manifolds, the fundamental group, representation theory of the symmetric group and generating functions in a skeletal way, touching only on content that we considered essential to our scope. Such background is not collected all together at the beginning, but is introduced at the moment when it is needed in the story, which we believe develops the exposition in a more organic way.

We made the choice of having exercises interspersed in the narration of the book, serving as an integral part of the exposition, rather than collecting exercises at the end of each section. The exercises are designed to develop familiarity with the concepts introduced, which is necessary before using the concepts in new ways. Exercises also appear in proofs, partly to avoid the excessive proliferation of parts of proofs that consist mostly in bookkeeping, but also to encourage the reader to be actively involved and test his/her understanding.

This book can and should be used differently by different readers, but we hope that, whether you are an instructor preparing a course, a student reading this independently, or something in between, you find this book a helpful guide through the first steps in this fascinating topic.

Although the main body of the text covers a lot of ground, this is really only the beginning of the story in Hurwitz theory. By nature, Hurwitz theory is interdisciplinary and is part of the basic toolkit in many areas of mathematics. In the appendices we offer a glimpse of what is beyond through a small number of essays by guest writers: active researchers in various areas of mathematics who use Hurwitz theory in their work. The scope of the appendices is to pique the reader’s interest; to leave them a bit dazed and confused, and with the desire to continue learning – which is the constant state of mind of any mathematician.

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