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Excerpt
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# 1

# From Complex Analysis to Riemann Surfaces

This chapter makes a quick and targeted incursion into the world of complex analysis, with the goal of presenting the ideas that historically led to the development of the notion of a Riemann Surface.

Differentiability for a function of one complex variable imposes considerably more structure than the analogous notion for functions of a real variable. Somewhat strangely, many of the remarkable properties of complex differentiable functions are natural consequences of a construction that somehow "leaves" the complex world: complex functions can be integrated along real paths, and the value of such integrals doesn't change if the path is continuously perturbed while fixing the endpoints.

This phenomenon leads to Cauchy's formula, which expresses a complex differentiable function as a path integral of yet another complex function. While this may seem a slightly bizarre thing to do, Cauchy's formula has a number of remarkable consequences. In particular it gives a differentiable expression for the local inverse to a complex differentiable function at a point where the derivative does not vanish.

When a differentiable $f$ function is not injective, obviously there does not exist a global inverse function. However, at any point where $f'$ doesn't vanish, one has multiple local inverse functions (or historically one said that the inverse of $f$ is a multivalued function) and, further, there is a natural way to view all these local inverses as part of a global function defined on a space which, around any point, "looks like" the complex numbers but globally may be different from $\mathbb{C}$. Such spaces are examples of Riemann Surfaces.

In this chapter, which is meant to illustrate how the concept of Riemann Surfaces was developed, we limit ourselves to exploring this picture for the power functions $w = z^k$ and their inverses (the $k$-th root "functions"). While this may seem unimpressive, Lemma 1.4.4 shows that the power functions, up to appropriate changes of variables, describe the behavior of any holomorphic function around a critical point – a point where the derivative vanishes.

Complex analysis is a beautiful and rich subject, and there is no way that we can do it justice in a handful of pages. We have made the choice of taking a path through the subject that gives a working understanding of a small selection of ideas that are important for the development of our story. We refer the reader interested in further reading to any textbook in complex analysis; for example, Conway (1978).

## 1.1 Differentiability

The definition of differentiability for functions of one complex variable is in complete analogy with the real variable case.

**Definition 1.1.1.** A function $f : \mathbb{C} \to \mathbb{C}$ is **differentiable** or **holomorphic** at a point $z_0 \in \mathbb{C}$ if and only if the following limit exists:

$$\lim_{|h| \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = L \in \mathbb{C}. \tag{1.1}$$

The complex number $L$ is the value of the derivative of $f$ at $z_0$, denoted by $f'(z_0)$. A function $f$ is differentiable on a domain $U \subseteq \mathbb{C}$ if it is differentiable at every point $u \in U$.

Because the complex numbers are two-dimensional over the real numbers, there are many ways for a complex variable $h$ to approach 0. Hence, the existence of the above limit imposes greater structure on functions of a complex variable. For instance, we have the following properties, which we prove in Section 1.3:

1. If $f : \mathbb{C} \to \mathbb{C}$ is differentiable in a neighborhood $U$ of $z_0$, then it is infinitely differentiable in $U$.
2. If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0$, then it is **analytic**, meaning that the Taylor expansion of $f$ at $z_0$ always converges to $f$ in a neighborhood of $z_0$.

The statements above are not true for functions of a real variable, as illustrated in the following exercise.

**Exercise 1.1.1.**

1. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ exists and is a continuous but not differentiable function.

2. Consider the function

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \mathrm{e}^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Show that $g$ is infinitely differentiable at 0 and that all derivatives vanish: $g^{(n)}(0) = 0$. This implies that the Taylor expansion of $g$ centered at 0 is identically 0 (and so the Taylor series does not converge to $g$ in a neighborhood of 0).

Writing complex numbers in Cartesian coordinates $z = x + iy$ for $x, y \in \mathbb{R}$, we may write a function $f(z)$ as $f(x, y) : \mathbb{R}^2 \to \mathbb{C}$. Identifying also the codomain of $f$ with $\mathbb{R}^2$, one has $f(x, y) = u(x, y) + iv(x, y)$ for functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. These are the real and imaginary parts, respectively, of $f$.

**Exercise 1.1.2.** Find the functions $u(x, y)$ and $v(x, y)$ associated with $f(z) = z^2$. Compute the derivative $f'(z)$ and find its real and imaginary parts.

**Theorem 1.1.2** (Cauchy–Riemann Equations)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open subset $U \subset \mathbb{C}$. Considering $f = u + iv$ as a real differentiable function on $\mathbb{R}^2$, then the following identities of partial derivatives hold on $U$:*

$$u_x = v_y, \qquad v_x = -u_y. \tag{1.2}$$

*Proof* One may restrict the difference quotient (1.1) to real paths approaching $z_0$. If $f$ is differentiable at $z_0 = x_0 + iy_0$, the limit is $f'(z_0)$ independently of the choice of path. We consider a vertical and horizontal path approaching $z_0$.

Letting $h$ approach zero along a vertical path gives (note that here $t \in \mathbb{R}$):

$$\begin{aligned} f'(z_0) &= \lim_{t \to 0} \frac{u(x_0, y_0 + t) + iv(x_0, y_0 + t) - (u(x_0, y_0) + iv(x_0, y_0))}{it} \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0). \end{aligned} \tag{1.3}$$

Similarly, letting $h$ approach zero horizontally yields

$$\begin{aligned} f'(z_0) &= \lim_{t \to 0} \frac{u(x_0 + t, y_0) + iv(x_0 + t, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{t} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0). \end{aligned} \tag{1.4}$$

Equating the real and imaginary parts from the two computations gives the result. ☐

The following corollary of the Cauchy–Riemann equations will be extremely important in our story.

**Corollary 1.1.3.** *Let $f$ be a non-constant holomorphic function. As a function from the real plane to itself, $f$ is orientation-preserving.*

*Sketch of proof* Intuitively, an orientation of the plane amounts to specifying the notions of "clockwise" and "counterclockwise". Formally, one defines an orientation as an equivalence class of bases of $\mathbb{R}^2$, where two bases are equivalent if the determinant of the change of basis matrix is positive. As a consequence, a function $f = u + iv : \mathbb{R}^2 \to \mathbb{R}^2$ is then said to be orientation-preserving if, on an open dense set of the domain of definition, the determinant of the Jacobian matrix

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is positive. If $f$ is complex differentiable, then the Cauchy–Riemann equations imply

$$\det J(f) = u_x v_y - v_x u_y = u_x^2 + v_x^2 \geq 0,$$

and since $f$ is not constant the inequality is strict on a dense open set. □

We conclude this section by stating without proof an important property of holomorphic functions. A proof is found, for example, in Conway (1978, Chapter IV, §7).

**Theorem 1.1.4** (Open Mapping Theorem)**.** *A non-constant holomorphic function $f$ is* **open***: if $U$ is an open subset of $\mathbb{C}$, then so is $f(U)$.*

## 1.2 Integration

Complex analytic functions can be integrated along paths in $\mathbb{C}$ (see Figure 1.1). For a path $\gamma : [a, b] \to \mathbb{C}$ define

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{1.5}$$

**Example 1.2.1.** We compute $\int_\gamma \frac{1}{z}dz$ for $\gamma$, a circle of radius $r$ centered at zero. We have $\gamma(t) = re^{2\pi it}$ and $\gamma'(t) = 2\pi ire^{2\pi it}$ for $t \in [0, 1]$. Then

$$\int_\gamma \frac{1}{z}dz = \int_0^1 \frac{1}{re^{2\pi it}} 2\pi ire^{2\pi it}dt = \int_0^1 2\pi idt = [2\pi it]_{t=0}^{t=1} = 2\pi i.$$

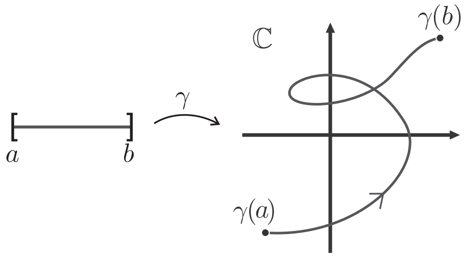


Figure 1.1 Picture of a path in $\mathbb{C}$

**Exercise 1.2.1.** Show that, for any integer $n \neq -1$, the integral $\int_\gamma z^n dz = 0$, where $\gamma$ is a circle of radius $r$ centered at zero.

A remarkable property of complex path integrals is that they are invariant under continuous deformations of the path. Intuitively, this means that we can wiggle the circle $\gamma$ in Example 1.2.1 to be any simple closed curve around the point 0 and still obtain the same result.

We will formalize and discuss extensively the notion of continuous deformation of paths (technically, **homotopy**) in Chapter 5. For now, if $\gamma, \eta : [a, b] \to U \subseteq \mathbb{C}$ are paths with the same endpoints (i.e. $\gamma(a) = \eta(a) = z_a$ and $\gamma(b) = \eta(b) = z_b$), a continuous deformation of $\gamma$ into $\eta$ is a continuous function $H : [a, b] \times [0, 1] \to U \subseteq \mathbb{C}$ satisfying $H(s, 0) = \gamma(s)$ and $H(s, 1) = \eta(s)$. We also ask that for every $t$, $H(a, t) = z_a$ and $H(b, t) = z_b$.

The idea is that at time $t = 0$ one has the path $\gamma(s)$, and as time $t$ flows from 0 to 1, the path $\gamma(s)$ continuously morphs into the path $\eta(s)$ while both endpoints stay fixed (see Figure 1.2).

**Theorem 1.2.2.** *Suppose that $\gamma, \eta : [a, b] \to U \subseteq \mathbb{C}$ are related by a continuous*[1] *deformation of paths. Then for any holomorphic function $f$ on $U$, we have*

$$\int_\gamma f(z)dz = \int_\eta f(z)dz.$$

*Proof* For any $t \in [0, 1]$ we integrate the function $f(z)$ along the path $H(s, t)$, obtaining the function $Int(t) = \int_{H(s,t)} f(z)dz$. Consider the derivative of $Int(t)$ with respect to $t$:

[1] We note that our proof requires the stronger condition that $H(s, t)$ has partial derivatives.

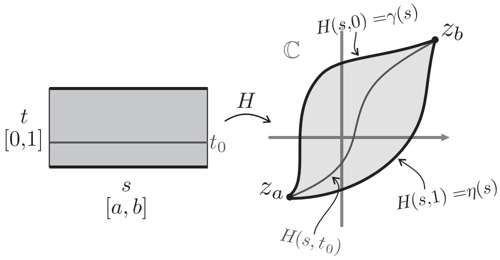


Figure 1.2 Schematic picture showing a homotopy of maps

$$\begin{aligned}\frac{d}{dt}Int(t) &= \frac{d}{dt}\int_a^b f(H(s,t))\frac{\partial H}{\partial s}(s,t)ds \\ &= \int_a^b \left( f'(H(s,t))\frac{\partial H}{\partial t}(s,t)\frac{\partial H}{\partial s}(s,t) + f(H(s,t))\frac{\partial^2 H}{\partial s \partial t}(s,t)\right) ds \\ &= \int_a^b \frac{d}{ds}\left[ f(H(s,t))\frac{\partial H}{\partial t}\right] ds = f(H(s,t))\left.\frac{\partial H}{\partial t}\right|_{s=a}^{s=b} = 0,\end{aligned}$$

since $H(a,t)$ and $H(b,t)$ are constant functions of $t$. Having derivative identically equal to 0, $Int(t)$ is a constant function and

$$\int_\gamma f(z)dz = Int(0) = Int(1) = \int_\eta f(z)dz.$$

□

**Corollary 1.2.3.** *Let $U$ be a simply connected region of $\mathbb{C}$ and $f$ a holomorphic function on $U$. For any closed path $\gamma$ whose image is inside $U$, $\oint_\gamma f(z)dz = 0$.*

*Sketch of proof* Let us recall that a path is said to be closed if its endpoints coincide. (The little circle on the integral sign is not strictly necessary, but it is a visual aid to emphasize that the integration is along a closed path.) The definition of being simply connected is essentially that any closed path may be continuously deformed to a constant path. The result now follows from Theorem 1.2.2 since integrating any function along a constant path yields 0 as a result. □

**Exercise 1.2.2.** Let $U$ be an open set in $\mathbb{C}$ and $f$ a holomorphic function on $U \setminus z_0$. For $j = 1, 2$, let $\gamma_j$ be a path parameterizing a circle centered at $z_0$ of radius $r_j$, oriented counterclockwise and completely contained in $U$. Show that:

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz.$$

In other words, the value of the path integral is independent of the radius of the circle.

## 1.3 Cauchy's Integral Formula and Consequences

From the invariance of path integrals under deformation of paths, one obtains a formula for a holomorphic function as a path integral.

**Theorem 1.3.1** (Cauchy's Integral Formula)**.** *Let $\gamma$ be a small loop around $z \in \mathbb{C}$ and $f(w)$ a holomorphic function in a neighborhood $U$ of $\gamma$. Then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z} dw. \tag{1.6}$$

A formal proof of formula (1.6) may be found in any complex analysis book; for example, Conway (1978, Chapter IV, §5). Let us briefly consider how we should think of this formula and why we should believe it. For any $z \in U$ we intend to describe the value of $f(z)$ as a path integral: at this point we consider $z$ a fixed complex number and the variable of integration is denoted by $w$. From Theorem 1.2.2 we may assume that $\gamma$ bounds a small disk around $z$, and from Exercise 1.2.2 we may let the radius of the disk shrink to 0 without altering the result of integration. Then the function $f(w)$ restricted to $\gamma$ tends to the (constant) complex number $f(z)$, whereas the path integral of $1/(w - z)$ is $2\pi i$, as seen in Example 1.2.1.

**Remark 1.3.2.** Cauchy's integral formula may seem baffling at first: if the goal is to understand the function $f$, why would one make any progress by replacing it with an integral function, which is a more complicated object, and furthermore an integral that involves $f$ itself as a part of the integrand? The answer is that (1.6) is not used to compute values of $f$, but to deduce properties of $f$ as a function by exploiting the nice formal properties of integrals. We now illustrate this idea by showing some remarkable consequences of Cauchy's formula.

The first remarkable consequence of Cauchy's integral formula is that any holomorphic function can be expressed, in a neighborhood of any point $z_0$, as a power series centered at $z_0$. In the string of equations that follows, we assume at every step that we restrict the domain to an appropriate neighborhood of $z_0$ as needed:

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0 + z_0 - z}dw = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}dw$$
$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0}\left(\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^n}\right)dw$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma \frac{f(w)\,dw}{(w-z_0)^{n+1}}\right)(z-z_0)^n. \tag{1.7}$$

Formula (1.7) implies that a holomorphic function is analytic: it is infinitely differentiable and the Taylor expansion about any point $z_0$ converges to the function in a neighborhood of $z_0$. Finally, it provides integral formulas for all derivatives of $f$:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}.$$

**Definition 1.3.3.** Given $n$, a positive integer, a complex function $f$ has a **pole of order** $n$ at the point $z_0 \in \mathbb{C}$ if $(z-z_0)^n f(z)$ is holomorphic at $z_0$ but $(z-z_0)^{n-1} f(z)$ isn't.

**Exercise 1.3.1.** Show that if $f$ has a pole of order $n$ at $z_0$, then it admits a **Laurent expansion** at $z_0$; i.e. in a neighborhood of $z_0$,

$$f(z) = \sum_{k=-n}^{\infty} a_k(z-z_0)^k,$$

with $a_{-n} \neq 0$.

**Definition 1.3.4.** Let $f$ have a pole of order $n$ at the point $z_0$. Then the **residue** of $f$ at $z_0$ is the $k = -1$ coefficient in the Laurent expansion of $f$ at $z_0$.

**Exercise 1.3.2.** Show that if $f$ has a pole of order 1 at $z_0$, then the residue of $f$ at $z_0$ can be computed as the following limit:

$$Res_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z). \tag{1.8}$$

**Exercise 1.3.3** (Residue theorem)**.** Let $\gamma : [a,b] \to U \subseteq \mathbb{C}$ be a simple closed path, bounding a region denoted $W$, containing the points $z_1, \dots, z_m$ (see Figure 1.3). Assume $f$ is holomorphic on $U \smallsetminus \{z_1, \dots, z_m\}$ and has polar singularities at the points $z_j$. Show that:

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{m} Res_{z=z_j} f(z). \tag{1.9}$$

## 1.4 Inverse Functions

An important result for us is the Inverse Function Theorem, which says that a holomorphic function admits a local inverse at any point where the derivative is not zero.

**Theorem 1.4.1** (Inverse Function Theorem)**.** *Let* $f : U \to \mathbb{C}$ *be a holomorphic function and* $z_0 \in U$ *such that* $f'(z_0) \neq 0$*. Then there exists a neighborhood* $V$ *of* $f(z_0)$ *and a holomorphic function* $g : V \to \mathbb{C}$ *such that* $z_0 \in g(V)$ *and for every* $z \in g(V)$*,* $g \circ f(z) = z$*.*

*Proof* Since $f'(z_0) \neq 0$ it is possible to restrict the domain of $f$ to an open neighborhood $U'$ of $z_0$ in such a way that $f$ is injective and $f'$ is never zero on $U'$ [2]. Since non-constant holomorphic functions are open functions, there exists a small ball $B_\delta$ centered at $f(z_0)$ with $B_\delta \subseteq f(U')$. Let $\gamma$ be a path parameterizing the boundary of $B_\delta$. If we let $V = B_\delta$ and restrict $f$ to $f^{-1}(V)$, we have a bijective function that admits a set theoretic inverse. We show that such a function is holomorphic by providing an integral formula for it. For any $w \in V$, define:

$$g(w) = \frac{1}{2\pi i} \oint_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta.$$

Let $z \in g(V)$ be such that $w = f(z)$. The integrand of $g(w)$ has a unique pole of order 1 at $\zeta = z$. Applying the Residue theorem:

$$g(w) = Res_{\zeta=z}\left(\frac{\zeta f'(\zeta))}{f(\zeta) - w}\right) = \lim_{\zeta \to z} (\zeta - z)\frac{\zeta f'(\zeta)}{f(\zeta) - w}.$$

Since $f(z) = w$, $\lim_{\zeta\to z} \frac{\zeta - z}{f(\zeta)-w} = \frac{1}{f'(z)}$, giving $g(w) = z$. □

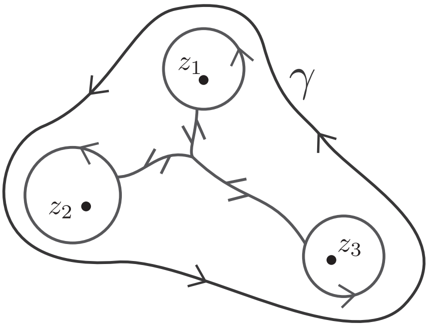


Figure 1.3 Idea behind proof of the Residue theorem

[2] For visual intuition, consider the analogous statement for a real-valued function.

**Remark 1.4.2.** The Inverse Function Theorem also holds for functions of many variables. A holomorphic function $F : \mathbb{C}^n \to \mathbb{C}^n$ is locally invertible at a point $\mathbf{z_0}$ (with a holomorphic local inverse around $F(\mathbf{z_0})$) if and only if $\det J(F)_{|\mathbf{z_0}} \neq 0$.

### 1.4.1 *k*-th Roots

Let $k \geq 1$ be an integer and consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $w = f(z) = z^k$.

**Exercise 1.4.1.** Show that, for any $w_0 \neq 0$, the inverse image $f^{-1}(w_0)$ has precisely $k$ elements. The only inverse image of 0 via $f$ is 0.

The derivative $f'(z) = kz^{k-1}$ only vanishes at $z = 0$; by the Inverse Function Theorem, $f$ is locally invertible at every $z \neq 0$. For any $w_0 \neq 0$ and $z_0$ such that $z_0^k = w_0$ there is a holomorphic function $f_{z_0}^{-1}$ defined in a neighborhood $U$ of $w_0$ such that $f_{z_0}^{-1}(w_0) = z_0$ and $f \circ f_{z_0}^{-1}(w) = w$ for all $w \in U$. Such a function is called a **branch** of the $k$-th root function $z = w^{1/k}$ near $w_0$.

A natural question is: how much can the domain of definition of a given branch of the $k$-th root be extended? Since $f_{z_0}^{-1}(w)$ provides a choice of a distinguished root for every point of $U$, one could imagine picking a point close to the boundary of $U$ and repeating the procedure to "enlarge" the domain of definition of $f_{z_0}^{-1}(w)$, perhaps eventually managing to "fill" all of $\mathbb{C} \setminus 0$ (see Figure 1.4). That this is not possible is illustrated by the following exercise.

**Exercise 1.4.2.** Let $z_0 = 1 \in \mathbb{C}$ identify a branch of $z = w^{\frac{1}{k}}$ near $w_0 = 1$. Consider the path $\gamma : [0, 1) \to \mathbb{C}$ given by $\gamma(t) = e^{2\pi i t}$. What is

$$\lim_{t \to 1} f_{z_0}^{-1} \circ \gamma(t)?$$

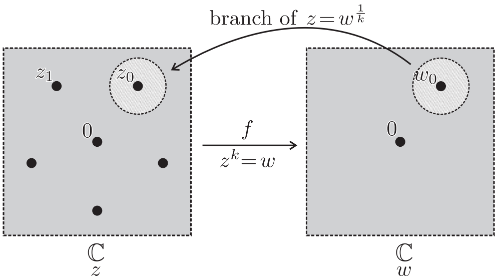


Figure 1.4 Domain and range of $w = z^k$ with branch of $k$-th root chosen