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Dispersive Partial Differential Equations

Wellposedness and Applications

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Preface

This book is intended for beginning graduate students in mathematics with some background in real and complex analysis who are interested in pursuing research in nonlinear dispersive partial differential equations (PDEs). This area has become exceedingly technical branching out into many different directions in recent decades. With this book, our aim is to provide a gentle introduction to the basic methods employed in this area in a self contained manner and in the setting of a few model equations. However, we should note that these methods are more generally applicable, and play a central role in modern research in nonlinear dispersive PDEs.

We designed this book having in mind a semester-long course in this area for advanced undergraduate and beginning graduate students. For that reason, we restricted the discussion to a few basic equations while providing complete details for each topic covered. We have also included many exercises that supplement and clarify the material that is discussed in the main text. After reading our book, a student should be able to read recent research papers in nonlinear dispersive PDEs and start making contributions.

There are several books, including Cazenave [28, 29, 30], Bourgain [20], Sulem–Sulem [138], Tao [143], and Linares–Ponce [105], which cover a large proportion of this area. In comparison, our book concentrates more on problems with periodic boundary conditions and aims to introduce the wellposedness techniques of model equations, such as the Korteweg de-Vries (KdV) and nonlinear Schrödinger (NLS) equations. The methods we describe also apply to various dispersive models and systems of dispersive equations, such as the fractional Schrödinger equation and the Zakharov system. In cases where the model equations are integrable, such as the periodic KdV and cubic NLS equations, alternative methods based on the symmetries and the structure of the equations have been developed. We refer the interested reader to Pöschel–Trubowitz [123], Kuksin [99], and Kappeler–Topalov [81] for complete integrability and inverse scattering techniques that extend some of the analytical results presented here. However, we should mention that we will not make use of any complete integrability methods in this book.

The KdV and NLS equations are the simplest models, combining the effects of dispersion and nonlinear interactions. The KdV equation describes very diverse physical phenomena, such as surface water waves in shallow water, propagation of ion-acoustic waves in cold plasma, and pressure waves in liquid-gas bubble mixture. In the case of shallow water, one normally does not work with the full water wave equation but uses approximate models to study the evolution. In particular, the KdV equation describes unidirectional small amplitude long waves on a fluid surface.

The NLS equation arises in a number of physical models in the theory of nonlinear optics. For example, it frequently appears as the leading approximation of the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium. It also arises in the description of Bose–Einstein condensation. Another equation we consider in this book is the fractional NLS equation, which is a basic model in the theory of fractional quantum mechanics. It is also used as a model describing charge transport in bio polymers like DNA.

The NLS equation, having a power nonlinearity, is easier to deal with in high regularity spaces by Sobolev embedding techniques. For lower regularity solutions on $\mathbb{R}^n$, Strichartz estimates are the main tools to establish wellposedness. On the other hand, in the case of the KdV equation, the derivative nonlinearity makes the problem more complicated. In fact, even the existence of smooth solutions requires more elaborate techniques. The situation is even more complicated for initial value problems on bounded domains, where the dispersion is weaker, and the wellposedness is harder to establish, especially in low regularity spaces.

In recent decades, a variety of techniques utilizing harmonic analysis methods were applied in conjunction with classical PDE tools to address these difficulties. Most of these techniques rely on time averaging via space-time norm estimates. Along these lines, we discuss Strichartz estimates, which is a very efficient method of establishing the wellposedness of dispersive PDEs with power type nonlinearities. We also discuss oscillatory integral techniques, which is based on the representation of the solution using the Fourier transform in the space variable. This technique is very efficient when dealing with equations with derivative nonlinearities. In addition, we present the restricted norm method using an anisotropic space-time Sobolev norm which takes into account the distance between the space-time Fourier supports of the linear and nonlinear solutions.

We now give a short summary of the contents of the book, which is divided into five chapters. In the first chapter, we recall without proof basic results from analysis that will be used throughout the text. Although we expect the reader to be familiar with basic harmonic analysis techniques, all the results we need in this book are outlined in this chapter.

In the second chapter, we concentrate on linear dispersive equations on the real line and on the torus. The methods are perturbative around the linear solution and the mapping properties of the linear propagator are extremely important in studying nonlinear counterparts. In particular, to find out which space is suitable in order to analyze the nonlinear solution, one needs to understand the decay and smoothing properties of the linear solution. We thus establish Strichartz estimates, Kato smoothing, and maximal function estimates for equations on the real line, and Strichartz estimates for equations on the torus. In this book, we make an effort to present various applications of the methods we discuss which are not found in other books in the area. One such application is the so-called Talbot effect for nonlinear dispersive PDEs on the torus. We finish Chapter 2 with a discussion of the Talbot effect for linear equations. This discussion is also useful for understanding the differences between the dynamics of dispersive PDEs on bounded and unbounded domains.

In the third chapter, we study basic wellposedness methods for the KdV equation on the torus and the real line, and the NLS equation on the torus. We start with the energy method based on parabolic regularization and the conservations laws of the equation. This method applies equally well to dispersive and nondispersive evolution equations, and it is a useful tool for studying smooth solutions. Then we discuss the oscillatory integral method of Kenig–Ponce–Vega, which uses the dispersive estimates established in Chapter 2. This method is useful mainly for the equations on $\mathbb{R}^n$. We continue with the restricted norm method of Bourgain. We then proceed to establish a version of the normal form transform, which we use to establish nonlinear smoothing and unconditional wellposedness results. We close this chapter with a thorough discussion of illposedness results.

In the fourth chapter, we study rough data global wellposedness and nonlinear smoothing of model dispersive equations. In particular, we present Bourgain's high–low decomposition method to establish global solutions when no a priori bounds are available. We also discuss the method of almost conserved quantities of Colliander–Keel–Staffilani–Takaoka–Tao, which can be considered as a refinement of the high–low decomposition method.

Finally, in the fifth chapter we present some applications of the techniques we developed in the previous chapters. More precisely, we study the growth bounds for higher order Sobolev norms, almost everywhere convergence to initial data for rough nonlinear solutions, the Talbot effect for nonlinear equations, and the existence and regularity of global attractors for dissipative and dispersive equations.

During the writing of this book the first author was partially supported by NSF grants DMS-1201872 and DMS-1501041. The second author was partially supported by the NSF grant DMS-0901222, the Simons Foundation grant #355523, and the University of Illinois Research Board grant RB-14054.

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December 2015

Notation

$\langle x\rangle$	$= \sqrt{1+\|x\|^2}$, $x \in \mathbb{R}^n$.
$B(x,r)$	The ball centered at x with radius r.
$A \lesssim B$	$A \le CB$, where $C > 0$ is an absolute constant.
$A \approx B$	$A \lesssim B$ and $B \lesssim A$.
$A \lesssim B^{s\pm}$	$A \lesssim B^{s\pm\epsilon}$ for any $\epsilon > 0$.
$A \ll B$	$A \le \frac{1}{C}B$, where C is a sufficiently large constant.
$A = O(B)$	$A \lesssim B$.
$A = o(B)$	$\lim \frac{A}{B} = 0$.
$\mathbb{R}$	The field of real numbers.
$\mathbb{T}$	The torus $\mathbb{R}/2\pi\mathbb{Z}$.
$\mathbb{C}$	The field of complex numbers.
$L^p(K)$	The Lebesgue spaces of measurable functions (for $K = \mathbb{T}$ or $\mathbb{R}$): $\left\{f : K \to \mathbb{C} : \|\|f\|\|_{L^p}^p := \int_K \|f\|^p < \infty\right\}$, $p \in [1,\infty)$, with the usual modification when $p = \infty$.
ℓ^p	$= \left\{a : \mathbb{Z} \to \mathbb{C} : \|\|a\|\|_{\ell^p}^p = \sum_{k\in\mathbb{Z}} \|a_k\|^p < \infty\right\}$, $p \in [1,\infty)$.
$\langle f, g\rangle_{L^2(K)}$	$= \int_K \overline{f(x)}g(x)dx$, $K = \mathbb{R}$ or $\mathbb{T}$.
$\mathcal{F}$	Fourier transform on $\mathbb{R}$: $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x}dx$, $\xi \in \mathbb{R}$, or Fourier series on $\mathbb{T}$: $\mathcal{F}f(k) = \widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$, $k \in \mathbb{Z}$.
$\mathcal{F}^{-1}$	Inverse Fourier transform on $\mathbb{R}$: $\mathcal{F}^{-1}\widehat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\xi x}dx$, $x \in \mathbb{R}$, or on the torus: $\mathcal{F}^{-1}\widehat{f}(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikx}$.

$f * g$	The convolution of f and g.
D^s	The multiplier operator with the multiplier $\lvert\xi\rvert^s$, $s \in \mathbb{C}$.
J^s	The multiplier operator with the multiplier $\langle\xi\rangle^s$, $s \in \mathbb{C}$.
$\lVert f\rVert_{H^s}$	$= \lVert J^s f\rVert_{L^2}$, $s \in \mathbb{R}$.
$\lVert f\rVert_{\dot{H}^s}$	$= \lVert D^s f\rVert_{L^2}$, $s \geq 0$.
$C_t^0 H_x^s$	The Banach space of H^s valued continuous functions with the norm $\sup_t \lVert u(t,\cdot)\rVert_{H^s}$
$\lVert T\rVert_{X\to Y}$	The operator norm of a bounded linear operator $T : X \to Y$ between Banach spaces X and Y.
X'	The dual of a topological vector space X.
T^*	The adjoint of an operator T.
$C^\infty(K)$	$= \{f : K \to \mathbb{C} : f \text{ is infinitely differentiable}\}, \quad K = \mathbb{R} \text{ or } \mathbb{T}$.
$C_0^\infty(\mathbb{R})$	$= \{f \in C^\infty(\mathbb{R}) : f \text{ is compactly supported}\}$.
$P_{m,n}(f)$	$= \left\lVert \langle x\rangle^m f^{(n)}(x)\right\rVert_{L^\infty}$.
$\mathcal{S}(\mathbb{R})$	Schwartz space: $\{f \in C^\infty(\mathbb{R}) : P_{m,n}(f) < \infty, m, n \geq 0\}$.
$\mathcal{D}(\mathbb{R})$	$= (C_0^\infty(\mathbb{R}))'$, the space of distributions.
$\mathcal{D}(\mathbb{T})$	$= (C^\infty(\mathbb{T}))'$, the space of periodic distributions.
$\mathcal{S}'(\mathbb{R})$	The space of tempered distributions.
$u(\phi)$	The action of the distribution u on the test function ϕ.
H	The Hilbert transform: $Hf(x) = \mathcal{F}^{-1}\left(i\,\mathrm{sign}(\cdot)\widehat{f}(\cdot)\right)(x)$.
$P_k f$	The Littlewood–Paley projection on to the frequencies $\approx 2^k$. We define $P_{\leq k}$, $P_{\geq k}$ similarly.

$P_N f$ The Littlewood–Paley projection on to the frequencies $\approx N$.

$B^s_{p,\infty}$ The Besov space defined by the norm:
$\|f\|_{B^s_{p,\infty}} := \sup_{j\geq 0} 2^{sj}\|P_j f\|_{L^p}$.

M Hardy–Littlewood maximal function:
$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy$.

W_t The propagator of the Airy equation, $W_t g = e^{-t\partial_{xxx}} g$.

W^γ_t The propagator of the weakly damped Airy equation:
$W^\gamma_t g = e^{-t\partial_{xxx}-t\gamma} g$.

$X^{s,b}$ The restricted norm space. In the case of the KdV equation, it is defined by the norm: $\|u\|_{X^{s,b}} = \left\|\widehat{u}(\tau,\xi)\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\right\|_{L^2_{\tau,\xi}}$.