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Excerpt
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Basic Equations

The flow of water in soil occurs through interconnected openings between the soil particles. The flow of water through the soil is erratic, and its velocity changes radically in space: the velocity is large in the small pores and small in the larger ones.

We do not need to determine the path that the water particles follow in their way through the soil for most engineering groundwater flow problems. It usually is sufficient to determine average velocities, average flow paths, the discharge flowing through a given area of soil, or the pressure distribution in the soil. We work, throughout this text, with averages and ignore the actual paths of flow. We use the term *rectilinear flow*, for example,when the average flow is in one direction.

The theory of groundwater flow is based on a law discovered by Henry Darcy [1856]. After the introduction of the basic concepts, we discuss the experiment performed by Darcy and present his law in its simplest form. We then present the generalized form of Darcy's law and the equation of continuity, and finish the chapter by combining these two equations into one governing equation for steady flow of a homogeneous fluid in a porous medium.

1.1 Basic Concepts

The basic quantities used to describe groundwater flow are velocity, discharge, pressure, and head. We discuss these quantities next.

1.1.1 The Specific Discharge

We define specific discharge as the volume of water flowing through a unit area of soil per unit time. The units of specific discharge are $[L^3/(L^2T)]$, or [L/T], and thus are the same as those of a velocity. Specific discharge sometimes is called discharge

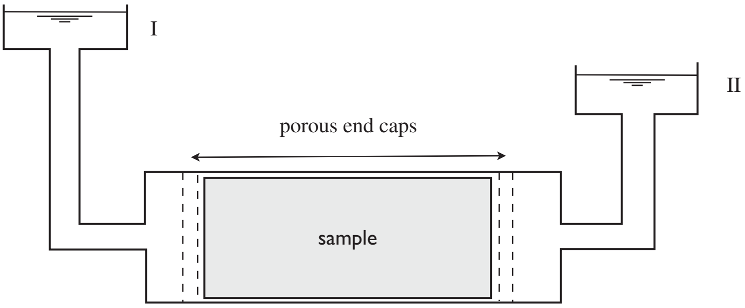

Figure 1.1 Schematic of Darcy's experiment.

velocity, but we use the term *specific discharge* to avoid confusion with a velocity. We represent the specific discharge by q.

The seepage velocity v is the average velocity at a point of the porous medium; it is the specific discharge divided by the area of voids present in a unit area of porous medium. If the porosity of the medium is n, then the area of voids per unit area is n and therefore

$$v = \frac{q}{n}. \tag{1.1}$$

Since n is always less than 1, v is always larger than q.

We illustrate the concepts of specific discharge and seepage velocity by considering water flowing through a cylindrical tube filled with sand, contained in the space between the end caps 1 and 2, as shown in Figure 1.1. This experimental setup is similar in principle to that used by Darcy to establish Darcy's law (Darcy [1856]). The cylinder is filled with water and is connected to two reservoirs, I and II, with different water levels. The water flows through the cylinder as a result of the difference in water levels. The level of reservoir II is controlled by overflow. We pour water into reservoir I in order to maintain its level. By measuring the rate at which water is poured into reservoir I, we determine the total amount, Q, of water flowing through the cylinder per unit time. The specific discharge is found by dividing Q by the cross-sectional area of the cylinder, A, i.e.,

$$q = \frac{Q}{A}. \tag{1.2}$$

We express the seepage velocity v with (1.1) as

$$v = \frac{q}{n} = \frac{Q}{nA}. \tag{1.3}$$

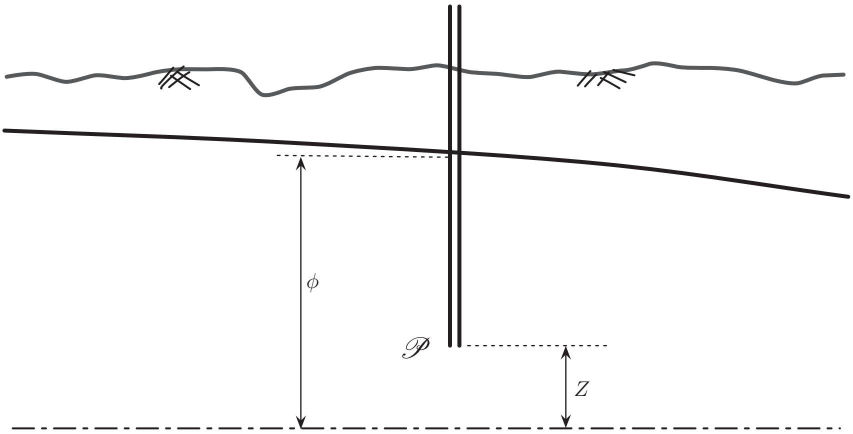

Figure 1.2 Definition of the head ϕ.

We obtain an expression for the time t for a water particle to travel through the sample from the seepage velocity v and the distance traveled, L:

$$vt = L \tag{1.4}$$

or

$$t = \frac{L}{v} = \frac{LnA}{Q}. \tag{1.5}$$

See (1.3).

1.1.2 Pressure and Head

The hydraulic head at a certain point $\mathscr{P}$ in a soil body is defined as the level to which the water rises in an open standpipe with its lower end at point $\mathscr{P}$ (see Figure 1.2). The *hydraulic head*, also simply called *head*,[1] is defined as a level and is measured with respect to a reference level or datum. We represent the hydraulic head by the letter ϕ. The units of ϕ are the units of length.

We find an expression for the pressure at point $\mathscr{P}$ from the weight of the water column above $\mathscr{P}$ in the standpipe. If the elevation of $\mathscr{P}$ above the reference level is Z [L], then the height of the water column above $\mathscr{P}$ is $\phi - Z$. Denoting the pressure as p [F/L^2], the density of water as ρ [M/L^3] and the acceleration of gravity as g [L/T^2], the pressure at $\mathscr{P}$ is

$$p = \rho g(\phi - Z). \tag{1.6}$$

[1] Other terms are in use as well, such as piezometric head and potentiometric head. We avoid using these terms; the word *piezometric* suggests that pressure is involved, and *potentiometric* suggests that the head is a potential, which it is not.

The elevation Z of point $\mathscr{P}$ above the reference level is known as the elevation head of point $\mathscr{P}$. The head, ϕ, can be expressed in terms of pressure, p, and elevation head, Z, by the use of (1.6) as follows:

$$\boxed{\phi = \frac{p}{\rho g} + Z} \tag{1.7}$$

The fraction $p/(\rho g)$, with the units of $[(F/L^2)/(F/L^3)] = [L]$ is the *pressure head*. We may express (1.7) as *(hydraulic) head equals pressure head plus elevation head*.

1.2 Darcy's Law

Darcy's law (Darcy [1856]) is an empirical relation for the specific discharge in terms of the head. The original form of this law is applicable to rectilinear flow of a homogeneous liquid only. A general form of Darcy's law exists; we present it after covering the case of rectilinear flow.

1.2.1 Rectilinear Flow

Darcy found that the amount of flow through a cylinder of sand of cross-sectional area A increases linearly with the difference in head at the ends of the sample (see Figure 1.3). The head at end cap 1 is ϕ_1. We see this from Figure 1.3: the pipe or hose connecting reservoir I to the sample can be viewed as a standpipe. Similarly, the head at end cap 2 is ϕ_2. Darcy's law for the experiment of Figure 1.3 is

$$Q = kA\frac{\phi_1 - \phi_2}{L}, \tag{1.8}$$

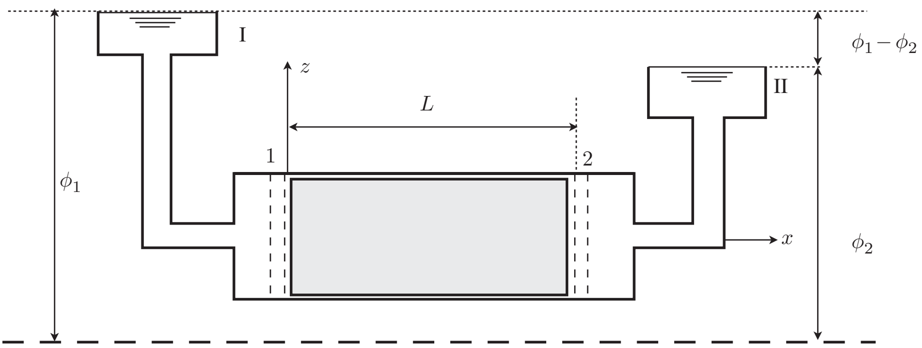

Figure 1.3 Darcy's experiment.

where Q is the discharge [L^3/T], A is the cross-sectional area of the sample [L^2], and L is the length of the sample [L]. The proportionality constant k is known as the hydraulic conductivity. It follows from (1.8) that the dimensions of k are those of a velocity, [L/T]. We sometimes use the term *resistance to flow*, borrowing this concept from electrokinetics. Resistance to flow is the inverse of hydraulic conductivity $(1/k)$.

We write (1.8) in terms of the specific discharge q, with $q = Q/A$,

$$q = k\frac{\phi_1 - \phi_2}{L}. \tag{1.9}$$

If we measure the head at various points inside the sample of Figure 1.3, we find that it varies linearly over the sample. Choosing a coordinate system with the x-axis running along the axis of the sample with the origin at end cap 1, we obtain the following expression for ϕ:

$$\phi = \phi_1 + \frac{\phi_2 - \phi_1}{L}x. \tag{1.10}$$

This equation represents the straight line from $x = 0$, $\phi = \phi_1$, to $x = L$, $\phi = \phi_2$ in Figure 1.4, where the head is plotted as a function of position over the sample. It follows from (1.10) that

$$\frac{d\phi}{dx} = \frac{\phi_2 - \phi_1}{L}, \tag{1.11}$$

so that we may write (1.9) as

$$q_x = -k\frac{d\phi}{dx}. \tag{1.12}$$

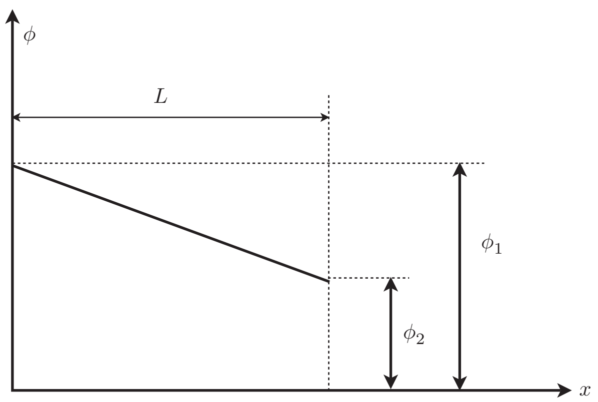

Figure 1.4 Linear variation of ϕ.

The index x in q_x is used to indicate that the specific discharge is in the x-direction. The derivative $d\phi/dx$ is known as the hydraulic gradient for flow in the x-direction.

1.2.2 Intrinsic Permeability

The hydraulic conductivity is a material constant, which depends on the properties of both the fluid and the soil. It is possible to define another constant, the (intrinsic) permeability κ, which depends only on the soil properties and is related to the hydraulic conductivity as

$$k = \frac{\kappa \rho g}{\mu}, \tag{1.13}$$

where μ is the dynamic viscosity [FT/L^2]. The dimension of κ is L^2. The intrinsic permeability is used primarily when the density or the viscosity of the fluid varies with position. In this text, however, only fluids with homogeneous properties are considered and therefore the classical hydraulic conductivity k is used. Values for k and κ are listed in Table 1.1 for some natural soils. An alternative unit for permeability is the Darcy, named after Henry Darcy. One Darcy, equal to 1000 μD (millidarcy), is equal to 0.9869233 μm^2.

1.2.3 Range of Validity of Darcy's Law

Darcy's law is restricted to specific discharges less than a certain critical value. The critical specific discharge depends on the grain size of the soil and the specific mass and the viscosity of the fluid. The criterion for assessing the validity of Darcy's

Table 1.1 *Permeabilities for some natural soils*

	k [m/s]	κ [m^2]
Clays	$< 10^{-9}$	$< 10^{-17}$
Sandy clays	$10^{-9} - 10^{-8}$	$10^{-16} - 10^{-15}$
Peat	$10^{-9} - 10^{-7}$	$10^{-16} - 10^{-14}$
Silt	$10^{-8} - 10^{-7}$	$10^{-15} - 10^{-14}$
Very fine sands	$10^{-6} - 10^{-5}$	$10^{-13} - 10^{-12}$
Fine sands	$10^{-5} - 10^{-4}$	$10^{-12} - 10^{-11}$
Coarse sands	$10^{-4} - 10^{-3}$	$10^{-11} - 10^{-10}$
Sand with gravel	$10^{-3} - 10^{-3}$	$10^{-10} - 10^{-9}$
Gravels	$> 10^{-2}$	$> 10^{-9}$

(Source: A. Verruijt, *Theory of Groundwater Flow*, © 1970, p. 10. Reprinted by permission of the author.)

law in a given case is expressed in terms of the Reynolds number Re, defined for groundwater flow as follows:

$$Re = \frac{D\rho}{\mu} q, \tag{1.14}$$

where D is the average grain diameter [L]. The Reynolds number is dimensionless. The range of validity of Darcy's law is defined by a relation obtained experimentally,

$$Re \leq 1. \tag{1.15}$$

If the Reynolds number is larger than 1, Darcy's law is not valid, and other, more complex, equations of motion must be used.

Darcy's law is valid for most cases of flow through soils. This is seen by substituting some average values for q, D, ρ, and μ in (1.14). The dynamic viscosity, μ, of water at a temperature of 10° C is about $1.3 * 10^{-3}$ N s/m^2 and ρ is about 10^3 kg/m^3. The average particle size of coarse sand is about $0.4 * 10^{-3}$ m in diameter. Substitution of these values in (1.14) yields

$$Re = (0.3 * 10^3)q. \tag{1.16}$$

This number is smaller than 1 if

$$q < 3.3 * 10^{-3} \text{ m/s} = 3.3 \text{ mm/s}. \tag{1.17}$$

This is a large value for the flow of groundwater. The hydraulic conductivity ranges from less than 10^{-9} m/s for clays to about 10^{-3} m/s for coarse sands. Furthermore, k is equal to the specific discharge occurring when the hydraulic gradient is 1, a large value. The specific discharge is the product of the hydraulic gradient and k, and is usually less than 10^{-3} m; Darcy's law indeed appears to be valid for most cases of flow through soils.

1.2.4 General Form of Darcy's Law

The flow is rarely rectilinear in practice, and neither the direction of flow nor the magnitude of the hydraulic gradient is known. The simple form (1.12) of Darcy's law, is not suitable for solving problems in practice; it is necessary to use a generalized form of Darcy's law, which gives a relation between the specific discharge vector and the hydraulic gradient. The direction of the specific discharge vector usually varies with position. The magnitude of this vector represents the amount of water flowing per unit time through a plane of unit area normal to the direction of flow. In three dimensions, the specific discharge vector has three components. With reference to a Cartesian coordinate system x, y, z, the three components of the specific discharge vector are represented as q_x, q_y, and q_z. The form of Darcy's law for three-dimensional flow through an isotropic porous medium is

$$\begin{aligned} q_x &= -k\frac{\partial \phi}{\partial x} \\ q_y &= -k\frac{\partial \phi}{\partial y} \\ q_z &= -k\frac{\partial \phi}{\partial z}. \end{aligned} \tag{1.18}$$

Because the three components of the specific discharge vector are proportional to minus the three components of the hydraulic gradient, with k as the proportionality factor, the specific discharge vector points in the direction opposite to the hydraulic gradient; groundwater flow occurs in the direction of decreasing head, hence the minus sign in Darcy's law. We sometimes represent the specific discharge vector with components (q_x, q_y, q_z) briefly as q_i. The index i then stands for x, y, or z. The partial derivatives $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial\phi/\partial z$ represent the three components of the hydraulic gradient. We may write the components of this vector as $\partial_x\phi, \partial_y\phi$, and $\partial_z\phi$, where the ∂ with the index stands for differentiation with respect to the coordinate represented by the index. The hydraulic gradient can then be written as

$$\partial_i\phi = \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right]. \tag{1.19}$$

The notation with indices is known as the indicial notation or tensor notation, and has the advantage of compactness. The three equations (1.18), for example, can be written as one,

$$q_i = -k\partial_i\phi. \tag{1.20}$$

Darcy's law (1.18) may be written in terms of pressure by the use of (1.6),

$$\begin{aligned} q_x &= -\frac{k}{\rho g}\frac{\partial p}{\partial x} \\ q_y &= -\frac{k}{\rho g}\frac{\partial p}{\partial y} \\ q_z &= -k - \frac{k}{\rho g}\frac{\partial p}{\partial z}, \end{aligned} \tag{1.21}$$

where the z-coordinate points vertically upward, so that $\partial Z/\partial z = 1$ (see Figure 1.2). Equations (1.18) and (1.21) are equivalent only if the density of the fluid is constant. Equation (1.18) is wrong in case ρ varies, as we demonstrate, following Verruijt [1970], by considering the case of groundwater of variable density at rest, so that

$q_x = q_y = q_z = 0$. We integrate (1.18) and use (1.7) to express ϕ in terms of the pressure,

$$\phi = \text{constant} = \frac{p}{\rho g} + Z. \tag{1.22}$$

Integration of (1.21) yields for this case

$$p = -\int_{Z_0}^{Z} \rho g dz, \tag{1.23}$$

where Z_0 is a reference level. Since (1.22) is not applicable to water of variable density at rest, and is obtained from (1.18), it follows that the latter equation cannot be used for cases of variable density, at least not with the definition (1.6) for ϕ. Equation (1.23), however, is correct and (1.21) is indeed valid for variable density.

Cases where variable density must be considered are not covered in this text, and we use Darcy's law in the form (1.18), with ϕ defined by (1.7).

1.2.5 Anisotropy

We assumed thus far that the hydraulic conductivity k is the same in all directions. In practice the soil often is layered; the hydraulic conductivity has different values in the directions parallel and normal to the layers. We call hydraulic conductivity anisotropic if its value depends on orientation. This is illustrated in Figure 1.5(a), where layers of sand are sandwiched between thin layers of clay. We consider the case in which there is no flow normal to the plane of drawing, the (x, y)-plane. We introduce Cartesian coordinates x^* and y^* such that the x^*-axis is parallel to the layers. It follows from Figure 1.5(b) that

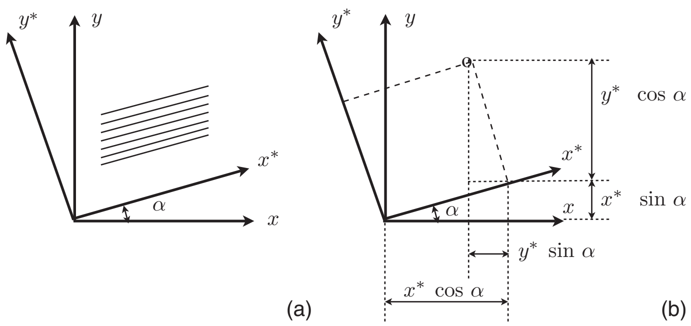

Figure 1.5 Anisotropic hydraulic conductivity.

$$
\begin{aligned}
x &= x^* \cos\alpha - y^* \sin\alpha \\
y &= x^* \sin\alpha + y^* \cos\alpha .
\end{aligned}
\tag{1.24}
$$

We write Darcy's law in terms of the x^*, y^*-coordinate system, denoting the components of the specific discharge vector in the x^* and y^* directions as q_x^* and q_y^*:

$$
\begin{aligned}
q_x^* &= -k_1 \frac{\partial \phi}{\partial x^*} \\
q_y^* &= -k_2 \frac{\partial \phi}{\partial y^*},
\end{aligned}
\tag{1.25}
$$

where k_1 and k_2 represent the values of the hydraulic conductivity in the directions parallel and normal to the layers, respectively. These directions are called the principal directions, and k_1 and k_2 the principal values of the hydraulic conductivity.

We write Darcy's law in terms of vector components in the x- and y-directions. The expressions for q_x and q_y in terms of q_x^* and q_y^* are similar to (1.24):

$$
\begin{aligned}
q_x &= q_x^* \cos\alpha - q_y^* \sin\alpha \\
q_y &= q_x^* \sin\alpha + q_y^* \cos\alpha .
\end{aligned}
\tag{1.26}
$$

We obtain from (1.25)

$$
\begin{aligned}
q_x &= -k_1 \frac{\partial \phi}{\partial x^*} \cos\alpha + k_2 \frac{\partial \phi}{\partial y^*} \sin\alpha \\
q_y &= -k_1 \frac{\partial \phi}{\partial x^*} \sin\alpha - k_2 \frac{\partial \phi}{\partial y^*} \cos\alpha .
\end{aligned}
\tag{1.27}
$$

By application of the chain rule we find

$$
\begin{aligned}
\frac{\partial \phi}{\partial x^*} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial x^*} = +\frac{\partial \phi}{\partial x}\cos\alpha + \frac{\partial \phi}{\partial y}\sin\alpha \\
\frac{\partial \phi}{\partial y^*} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial y^*} = -\frac{\partial \phi}{\partial x}\sin\alpha + \frac{\partial \phi}{\partial y}\cos\alpha ,
\end{aligned}
\tag{1.28}
$$

where the partial derivatives $\partial x/\partial x^*$, $\partial y/\partial x^*$, $\partial x/\partial y^*$, and $\partial y/\partial y^*$ are obtained by differentiating (1.24). Combining (1.27) and (1.28) we obtain Darcy's law for anisotropic hydraulic conductivity:

$$
\begin{aligned}
q_x &= -k_{xx} \frac{\partial \phi}{\partial x} - k_{xy} \frac{\partial \phi}{\partial y} \\
q_y &= -k_{yx} \frac{\partial \phi}{\partial x} - k_{yy} \frac{\partial \phi}{\partial y},
\end{aligned}
\tag{1.29}
$$