Noncommutative Mathematics for Quantum Systems

Noncommutative mathematics is a significant new trend of mathematics in the twentieth century. Initially motivated by the formulation and development of quantum physics due to Heisenberg and von Neumann, the idea of 'making a theory noncommutative' has been extended to many areas of pure and applied mathematics. An example is quantum probability, describing the probabilistic aspects of quantum mechanics. The generalization from classical to quantum happens here in two steps: first the theory is reformulated in terms of algebras of functions on probability spaces, then the commutativity condition is dropped.

This book focuses on two current areas of noncommutative mathematics: quantum probability and quantum dynamical systems.

The first part of the book provides an introduction to quantum probability and quantum Lévy processes. It provides introduction to the notion of independence in quantum probability. The theory of quantum stochastic processes with independent and stationary increments is also highlighted. The second part provides an introduction to quantum dynamical systems. It focuses on analogies with fundamental problems studied in classical dynamics.

The text underlines the balance between two crucial aspects of noncommutative mathematics. On one hand the desire to build an extension of the classical theory provides new, original ways to understand well-known 'commutative' results and on the other hand the richness of the quantum mathematical world presents completely novel phenomena, never encountered in the classical setting. This text will be useful to students and researchers in noncommutative probability, mathematical physics and operator algebras.

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We would like to dedicate this book to Marysia,
who was 11 days old when the Bangalore school began.

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Preface

This monograph arose from the lectures delivered by the authors during the graduate school, being a part of the meeting 'Recent advances in Operator Theory and Operator Algebras', which took place between 31 December 2012 and 12 January 2013 in the Indian Statistical Institute in Bangalore. We are very grateful to Rajarama Bhat, Tirthankar Bhattacharyya and Jaydeb Sarkar for organizing the meeting, inviting us to speak and providing excellent working conditions during our stay at ISI, and the audience of the school for their active participation in our courses. The authors are strong believers in the usefulness of organizing graduate schools for mathematical students – in fact we first met during an analogous event in Greifswald (Germany) in 2003, respectively, as a lecturer and a participating PhD student, and have collaborated ever since.

Our lectures during the Bangalore school treated notions of independence and quantum Lévy processes in quantum probability (by the first named author) and quantum processes understood as noncommutative incarnations of classical dynamical systems (by the second named author). Both these topics, born from necessity to incorporate quantum models into mathematical approaches to study physical systems, have now become very active, broad areas of modern mathematical research. The monograph consists of two chapters: 'Independence and Lévy processes in quantum probability', authored by Uwe Franz, and 'Quantum dynamical systems from the point of view of noncommutative mathematics', authored by Adam Skalski. These can be read independently, but we believe that there is an added value in placing them together; not in the least because they present alternative approaches to the noncommutative/quantum generalizations of classical concepts and results. The monograph is essentially self-contained, with several references to both sources of necessary background and to current research literature, and should thus form an appropriate entrance point for graduate students interested in the area.

We would like to express our gratitude to Rajarama Bhat and Gadadhar Misra for encouraging us to transform the lecture notes into the monograph form, and facilitating the contacts with the publisher, Cambridge University Press India, whose assistance during the editing process is also gratefully acknowledged.

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The Bangalore Center of the Indian Statistical Institute (ISI) has been holding a Workshop cum Conference on Operator Theory and Operator Algebras almost every alternate year for the last few decades. It has been growing in popularity. The name of the meeting has now stabilized to Recent Advances in Operator Theory and Operator Algebras (OTOA). More details about these workshops can be found at: http://www.isibang.ac.in/~statmath/conferences/conferences.html

In 2012–13, OTOA was organized by B.V. Rajarama Bhat, Tirthankar Bhattacharyya and Jaydeb Sarkar and was sponsored by ISI and the National Center for Mathematics (India). The main speakers in the workshop were Uwe Franz, K.R. Parthasarathy, Adam Skalski and Brett D. Wick.

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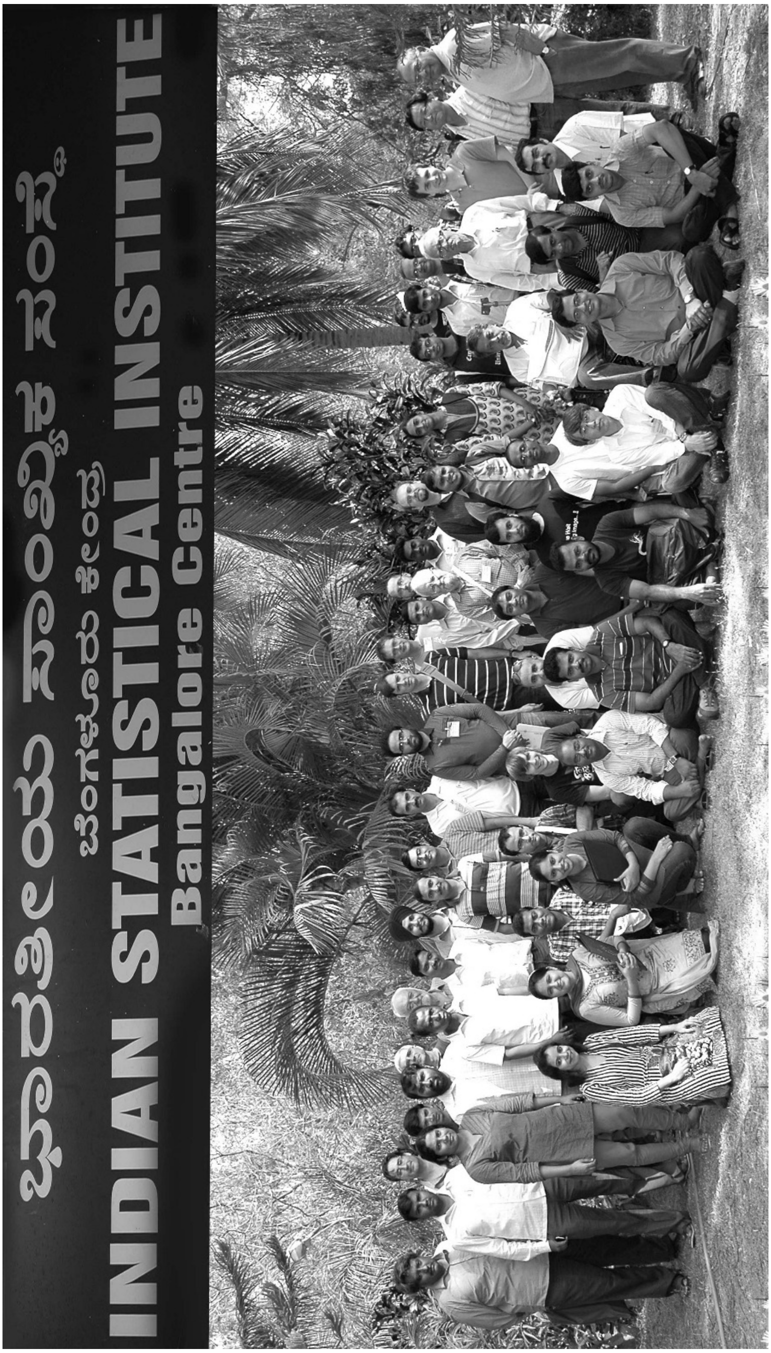

Conference photo of the meeting
"Recent advances in Operator Theory and Operator Algebras 2012" ISI, Bangalore, Dec. 31, 2012 - Jan. 11, 2013

Introduction

Noncommutative mathematics might be considered as one of the major new trends of mathematics that arose in the twentieth century. Initially motivated by the formulation and development of quantum physics due to Heisenberg and von Neumann, the idea of 'making a theory noncommutative' has been since extended to many areas of pure and applied mathematics and turned out to be extremely fruitful.

In 1925, Heisenberg suggested replacing the observables of a quantum system, such as its position Q and momentum P, by noncommutative quantities, most commonly realised as operators on a Hilbert space. This idea allowed the correct prediction of the results of the experiments of quantum physics; for example, it led to a description of the observed spectra of atoms. It has also led to a cornucopia of new mathematics, in particular to the theory of von Neumann algebras and C^*-algebras developed, respectively, by Murray and von Neumann, and by Gelfand and Naimark.

The general pattern of the noncommutative mathematics is the following: take a 'classical' mathematical theory, say topology, measure theory, differential geometry, or group theory, and reformulate it in terms of algebras. These algebras are algebras of functions on the spaces appearing in the 'classical' theory, that is, continuous functions (with values in $\mathbb{C}$) on a topological space, measurable functions on a measure space, smooth functions on a manifold, and so on. They inherit some additional structure, for example, an involution, a norm or topology, a coproduct, from the space on which they are defined. Finally, these algebras are always commutative. Axiomatizing the additional structure and dropping the commutativity condition one arrives at a noncommutative generalization of the original theory. In this way the theory of von Neumann algebras or C^*-algebras can be viewed, respectively, as noncommutative measure theory and noncommutative topology. The theory of Hopf algebras can be viewed as noncommutative group theory (here a warning is in place: the term 'noncommutative' refers to the function algebra, not to the group multiplication, which can also be noncommutative in classical 'commutative' group theory).

The procedure of making a theory noncommutative is not canonical, the choice of the axioms of the new noncommutative theory is not unique and each time has to be adapted to the classical theory to which it is applied. Because the idea originated from quantum physics, this procedure is also referred to as quantization and the new theory is labeled a 'quantum' theory. A good noncommutative theory should allow extending the central results of the classical theory and it should contain the classical theory in a clear way. A good example for the latter is the Gelfand functor, which defines an equivalence between the category of locally compact topological spaces and the category of commutative C^*-algebras.

The two lectures in this volume aim to present the rich new mathematics that was discovered in this way. We will concentrate on two fields of noncommutative mathematics: quantum probability, presented in the first chapter of this monograph, 'Independence and Lévy processes in quantum probability' (written by Uwe Franz), and quantum dynamical systems, treated in the second chapter, 'Quantum dynamical systems from the point of view of noncommutative mathematics' (written by Adam Skalski).

Quantum probability is a generalization of both classical probability theory and quantum mechanics that allows to describe the probabilistic aspects of quantum mechanics. 'Classical' probability spaces $(\Omega, \mathcal{A}, P)$ are replaced by the pairs $(L^\infty(\Omega), E(\cdot) = \int_\Omega \cdot \mathrm{d}P)$ consisting of the commutative von Neumann algebra of bounded random variables and the expectation functional. Then, the commutativity condition is dropped. In this way we arrive at the notion of a (von Neumann) algebraic probability space (N, Φ) consisting of a von Neumann algebra N and a normal (faithful tracial) state Φ. As we have seen this includes classical probability spaces in the form $(L^\infty(\Omega), E)$, however, also quantum mechanical systems modeled by a Hilbert space H and a pure state $\psi \in H$ (or a mixed state $\rho \in S(H)$), if we take $N = B(H)$ and Φ the state defined by $\Phi(X) = \langle \psi, X\psi \rangle$ (or $\Phi(X) = \mathrm{tr}(\rho X)$ for $X \in B(H)$).

We explain the setting of quantum probability in more detail in the second section of Franz' lecture. In the third section we discuss a version of the EPR experiment and a theorem by Kochen and Specker to explain that physics probably forces us to use quantum probability to describe our world at the quantum level.

Independence and Lévy processes in quantum probability. A striking feature of quantum probability (or noncommutative probability) is the existence of several notions of independence. This is the central topic of the remaining sections of Franz' lecture. We want to study the theory of the fundamental 'noises' in quantum probability. By a fundamental ‘noise’ we shall mean a quantum stochastic process with independent and stationary increments. These are generalizations of Lévy processes from classical probability, well-known special cases are Brownian motion and the Poisson process. Most other stochastic processes can be constructed from such processes, which explains why they play such a prominent role in applications.

In the fourth section of Franz’ lecture we briefly review the theory of Lévy processes in classical probability. In the fifth section we treat the case of tensor independence, which is the most natural generalization to quantum probability spaces of the classical notion of independence. This notion also corresponds to the notion of independent quantum observables used by physics. We give an introduction to Schürmann’s theory of Lévy processes on involutive bialgebras.

In the sixth section we develop this theory further under the assumption that the involutive bialgebra belongs to a compact quantum group in the sense of Woronowicz. These bialgebras have a richer structure and it is interesting to study Lévy processes that satisfy natural compatibility conditions with respect to this structure.

In the seventh section we introduce several other notions of independence that can be used in quantum probability. These are freeness, boolean independence, and monotone independence. For all these independences we can define convolutions. We show how these convolutions can be computed for probability measure on the real line or the unit circle using their Cauchy-Stieltjes transforms.

Next we present a classification by Muraki that states that these are the only ‘nice’ or ‘universal’ notions of independence. More precisely, they are the only possible notions on the category of noncommutative algebraic probability spaces that are based on an associative functorial product, as we shall see in the eighth section of Franz’ lecture.

Finally, in the last section of Franz’ lecture we study Lévy processes whose increments are independent in the sense of these notions. They are defined on dual groups, a kind of algebras similar to bialgebras, but with the usual tensor product of associative algebras replaced by their free product.

Dynamical systems. Skalski’s lecture presents several examples of how the theory of dynamical systems can be generalized to a noncommutative theory. The main focus is put on two building blocks of modern abstract theory of dynamical systems: entropy and ergodic theorems. The first section of the lecture introduces the concepts and examples of quantum topological dynamical systems (understood as endomorphisms of C^*-algebras) and develops systematically the analogies with classical topological spaces and classical dynamics. The three following sections concern the most successful generalization of the notion of topological entropy to quantum dynamical systems: Voiculescu entropy. Again several examples are given, including the computation of Voiculescu entropy of the shift endomorphism on Cuntz algebras. Here also some permanence properties of Voiculescu entropy are established and the role of classical subsystems of a given quantum dynamical system is discussed. Finally in sections 5 and 6 attention is turned toward quantum "measurable" dynamical systems, understood as normal endomorphisms of von Neumann algebras. Both mean and individual ergodic theorems are treated in that context, with the latter requiring a particularly novel (in comparison with the classical set-up) approach: the notion of almost everywhere convergence, which a priori requires a concept of 'points' in the space under investigation, is replaced by the almost uniform convergence, motivated by Egorov theorem.

Throughout both lectures the balance between two crucial aspects of noncommutative mathematics is underlined: on one hand the desire to build an extension of the classical theory often necessitates providing new, original ways to obtain and understand well-known "commutative" results; on the other hand the richness of the quantum mathematical world presents completely novel phenomena, never encountered in the classical setting. We hope this interplay will enchant our readers in the way similar to how it never ceases to amaze ourselves. If QP were always equal to PQ, the world would be infinitely more boring!