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Independence and Lévy Processes in Quantum Probability

1.1 Introduction

Quantum probability is a generalization of both classical probability theory and quantum mechanics that allows to describe the probabilistic aspects of quantum mechanics. This generalization is formulated in two steps. First, the theory is reformulated in terms of algebras of functions on probability spaces. Therefore, the notion of a probability space $(\Omega, \mathcal{F}, P)$ is replaced by the pair $(L^\infty(\Omega), E(\cdot) = \int_\Omega \cdot \, \mathrm{d}P)$ consisting of the commutative von Neumann algebra of bounded random variables and the expectation functional. Then, the commutativity condition is dropped. In this way we arrive at the notion of a (von Neumann) algebraic probability space (N, Φ) consisting of a von Neumann algebra N and a normal (faithful tracial) state Φ. As we have seen this includes classical probability spaces in the form $(L^\infty(\Omega), E)$, it also includes quantum mechanical systems modeled by a Hilbert space H and a pure state $\psi \in H$ (or a mixed state $\rho \in S(H)$), if we take $N = B(H)$ and Φ the state defined by $\Phi(X) = \langle \psi, X\psi \rangle$ (or $\Phi(X) = \mathrm{tr}(\rho X)$) for $X \in B(H)$. Note that in this course we shall relax the conditions on N and Φ and work with involutive algebras and positive normalized functionals, that is, $*$-algebraic probability spaces.

A striking feature of quantum probability (also called noncommutative probability) is the existence of several notions of independence. This is the starting point of this course, which

intends to give an introduction to the theory of quantum stochastic processes with independent increments.

However, before we come to these processes, we will give a general introduction to quantum probability. In Section 1.2, we recall the basic definitions of quantum probability and discuss some fundamental differences between classical probability and quantum probability. In Section 1.3 we address the question 'Why do we need Quantum Probability?' We discuss the EPR experiment and the Kochen–Specker Theorem, which show that we cannot model quantum physics with classical probability spaces because the values of observable quantities do not exist unless we specify which measurement we will carry out and which quantities we will determine. In this sense quantum physics requires a more radical description of chance. As so far all experiments have confirmed the sometimes counterintuitive predictions of quantum physics, it follows that quantum probability is necessary to describe the real world at the microscopic level.

For the rest of the course we choose to focus on models that are stationary and have certain independence properties.

In Section 1.4, we recall the basic theory of stochastic processes with independent and stationary increments in classical probability. The marginal distributions of such processes are infinitely divisible, see Definition 1.4.1, and form convolution semigroups, see Definition 1.4.2. We recall several classification results for infinitely divisible distributions and convolution semigroups.

In Section 1.5, we start with an important class of quantum Lévy processes, that is, quantum stochastic processes with independent and stationary increments, namely those defined on involutive bialgebras. Involutive bialgebras are involutive algebras equipped with an algebra homomorphism $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ from the algebra into its tensor product satisfying several conditions. This map allows to compose random variables and to define a notion of increments. The notion of independence that is used for this class of quantum Lévy processes is tensor independence, which carries its name because it is based on the tensor product of functionals and algebras. It generalizes the notion of stochastic independence used in classical probability and corresponds to the notion of independent observables in quantum

physics. The basic theory of these processes is owing to Schürmann, *cf.* [Sch93].

Recently a richer, more analytic theory of quantum Lévy processes, defined on Woronowicz' compact quantum groups [Wor87a, Wor98], has been initiated, *cf.* [CFK14]. While usually defined in the C*-algebraic setting (see Definition 1.6.1), these quantum groups can also be viewed as a special class of involutive bialgebras, sometimes also called CQG algebras. CQG algebras are involutive Hopf algebras canonically associated to Woronowicz' compact quantum groups, *cf.* [Wor87a, Wor98]. They have a richer structure, in particular, an antipode and a Haar state, which satisfies a KMS property. This allows to formulate properties of a Lévy process, which guarantee that the Markov semigroup can be extended to a C*- and a von Neumann algebra, and to the associated noncommutative L_p spaces. Cipriani, Franz, and Kula have used this additional structure to apply the theory of noncommutative Markov processes and noncommutative Dirichlet forms to Lévy processes on CQG algebras, see [CFK14]. In Section 1.6 we give an introduction to compact quantum groups and show that Lévy processes on compact quantum groups are in one-to-one correspondence with time- and space-homogeneous Markov semigroups, see Theorem 1.6.6.

In noncommutative probability there exist new, truly noncommutative notions of independence that have no counterpart in classical probability. Schürmann [Sch95b] has shown that it is possible to define quantum Lévy processes with increments that are independent in the sense of these new notions of independence, if one replaces the tensor product in the theory of bialgebras by the free product of algebras. In the last three sections of this course we will give an introduction to the quantum Lévy processes obtained in this way.

In Section 1.7, we give a first introduction to these so-called universal notions of independence. We define free, monotone, and boolean independence for subalgebras of a quantum probability space and study the convolutions associated to these independences for probability measures on the real line, the positive half-line, and the unit circle.

The universal independences are based on associative universal products of algebraic probability spaces. In Section 1.8, we study the independences from the point of view of products of

algebraic probability spaces and review their classifications. The results in this section are mostly owing to Ben Ghorbal and Schürmann [BGS02], and to Muraki [Mur02, Mur03].

In Section 1.9, we introduce quantum Lévy processes for universal independences. For this purpose we also introduce dual semigroups and dual groups, which are the counterparts of bialgebras and Hopf algebras. They can be obtained from bialgebras and Hopf algebras if one replaces in their definitions the tensor product by the free product of algebras, see also [Voi87, Zha91].

In Section 1.10, we close this course with a list of interesting research topics and open questions.

1.2 What is Quantum Probability?

Let us start with the most fundamental definition in quantum (or noncommutative) probability.

Definition 1.2.1 A *quantum probability space* is a pair (A, φ) consisting of a von Neumann algebra A and a normal state $\varphi : A \to \mathbb{C}$.

Remark 1.2.2 The conditions on the pair (A, φ) can be varied, depending on the applications we have in mind. In the main part of these lecture notes we will work with **-algebraic probability spaces*, which are pairs (A, φ) consisting of a unital $*$-algebra A and a normalized positive functional $\varphi : A \to \mathbb{C}$.

For a definition of a von Neumann algebra and normal functionals, and some motivation for their appearance in this context refer to Section 2.5.2 of the lecture of Adam Skalski in this volume.

Before we try to motivate the definition of a quantum probability space, let us recall the definition of a probability space given in classical probability theory.

Definition 1.2.3 A *'classical' probability space* is a triple $(\Omega, \mathcal{F}, P)$, where

- Ω is a set, the *sample space*, the set of all possible outcomes.
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra, the set of *events*.
- $P : \mathcal{F} \to [0, 1]$ is a probability measure, it assigns to each event its *probability*.

This description of randomness is based on the idea that randomness is because of a lack of information. If we know which $\omega \in \Omega$ is realized, then there is no randomness, and we know which outcome is realized for all possible experiments. However, in general this is not the case and therefore we want to work with all possible outcomes, and random variables which that are functions of these possible outcomes.

The following example shows that — in a certain sense — quantum probability contains classical probability as a special case.

Example 1.2.4 (Classical $\subseteq$ Quantum) To a classical probability space $(\Omega, \mathcal{F}, P)$ we can associate a quantum probability space (A, φ), take

- $A = L^\infty(\Omega, \mathcal{F}, P)$, the algebra of bounded measurable functions $f : \Omega \to \mathbb{C}$, called the algebra of *random variables* or *observables*.
- $\varphi : A \ni f \mapsto E(f) = \int_\Omega f \mathrm{d}P$, which assigns to each random variable/observable its expected value.

Then A is commutative and $(\Omega, \mathcal{F}, P)$ and (A, φ) are essentially equivalent (by the spectral theorem).

However, quantum probability is more general than classical probability. This additional generality is necessary to treat classical probability theory and the probabilistic structure of quantum mechanics in a common theory.

The following example is motivated by quantum mechanics.

Example 1.2.5 (Quantum mechanics) Let H be a Hilbert space, with a unit vector $\psi \in H$ (or a density matrix $\rho \in B(H)$). Then the quantum probability space associated to (H, ψ) (or (H, ρ)) is given by

- $A = B(H)$, the algebra bounded linear operators $X : H \to H$. Self-adjoint (or normal) operators can be considered as *quantum random variables* or *observables*.
- $\varphi : B(H) \ni X \mapsto \varphi(X) = \langle \psi, X\psi \rangle$, where $\psi \in H$ is a unit vector, or, more generally, $\varphi(X) = \mathrm{tr}(\rho X)$, where ρ is a density matrix.

Note that in this book inner products are always linear on the *right* side.

States of the form $\varphi(X) = \langle \psi, X\psi \rangle$ are called *pure states* or *vector states*. Note that unit vectors $\psi \in H$ and $\psi' = e^{i\phi}\psi$ that differ only by a phase and the orthogonal projection $P_\psi = |\psi\rangle\langle\psi| : u \mapsto \langle \psi, u\rangle\psi$ onto the subspace $\mathbb{C}\psi$ spanned by those vectors all define the same state. States of the form $\varphi(X) = \mathrm{tr}(\rho X)$ are called *mixed states*, if ρ is not a rank-one projection.

Is 'quantum randomness' different from 'classical randomness'? To discuss this question let us briefly recall how the quantum probability space presented above is used to model experiments in quantum mechanics.

For simplicity let us suppose that H is a finite dimensional complex Hilbert space.

Theorem 1.2.6 *(Spectral theorem) If $X \in B(H)$ is an observable (that is, a self-adjoint operator = hermitian matrix), then it can be written as*

$$X = \sum_{\lambda \in \sigma(X)} \lambda E_\lambda$$

where $\sigma(X)$ denotes the spectrum of X (= set of eigenvalues) and E_λ the orthogonal projection onto the eigenspace of X associated to the eigenvalue λ.

Physicists associate to the observables of a quantum mechanical system, like the position or momentum of a particle, the spin of an electron, or the polarization of a photon, a self-adjoint operator on some Hilbert space H. The state of the quantum mechanical system is described by a state on the algebra $B(H)$. This state is often given in the form of a density matrix, that is, a positive operator $\rho \in B(H)$ with trace equal to one. The special case where ρ is the orthogonal projection $\rho = |\psi\rangle\langle\psi|$ onto a unit vector $\psi \in H$ corresponds to a pure state and we call ψ its state vector. Note that we will freely switch between the various mathematical descriptions, that is, state vectors, density matrices, and states (in the sense of unital positive linear functionals) for the state of a quantum system.

Von Neumann's 'Collapse' Postulate:
A measurement of an observable X with spectral decomposition

$$X = \sum_{\lambda \in \sigma(X)} \lambda E_\lambda$$

on a quantum system in the state ρ can only yield values that belong to the spectrum of X. A value $\lambda \in \sigma(X)$ occurs with probability

$$p_\lambda = \mathrm{tr}(\rho E_\lambda)$$

where tr denotes the trace. If the observed value is λ, then the state 'collapses' to

$$\tilde{\rho}_\lambda = \frac{E_\lambda \rho E_\lambda}{\mathrm{tr}(\rho E_\lambda)},$$

that is, the state of the quantum system after the measurement is described by the density matrix $\tilde{\rho}_\lambda$.

As each $\lambda \in \sigma(X)$ occurs with probability

$$\rho_\lambda = P_\rho(X = \lambda) = \mathrm{tr}(\rho E_\lambda),$$

we get

$$E_\rho(X) = \sum_{\lambda \in \sigma(X)} \lambda p_\lambda = \mathrm{tr}(\rho X)$$

for the expectation and

$$\mathrm{Var}_\rho(X) = E_\rho\big((X - E_\rho(X))^2\big) = \mathrm{tr}(\rho X^2) - \big(\mathrm{tr}(\rho X)\big)^2$$

for the variance of the observable X in the state ρ.

The simplest experiments in quantum mechanics can produce only two possible outcomes, like the measurement of the spin of a fermion (like, for example the electron) in a fixed direction in a Stern-Gerlach-type experiment, or sending a single photon through a polarization filter. Such experiments are described by the two-dimensional Hilbert space $\mathbb{C}^2$.

Example 1.2.7 (Spin of a spin-$\frac{1}{2}$ particle or polarization of a photon) Consider $H = \mathbb{C}^2$. As vectors that differ only by a phase define the same state, we can assume that the first component of a state vector in $\mathbb{C}^2$ is not negative. Therefore, the most general state vector is of the form

$$u(\theta, \phi) = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

with $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, and $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$ form an orthonormal basis for $\mathbb{C}^2$ (for example, corresponding to the states 'spin up' and 'spin down').

Note that we used here the bra-ket notation, which is standard in quantum mechanics and also frequently used in related fields such as quantum probability and quantum information. Hilbert space vectors are denoted by so-called 'ket's' $|\text{label}\rangle$, linear functionals on the Hilbert space by 'bra's' $\langle\text{label}|$, and rank one operators by $|\text{label 1}\rangle\langle\text{label 2}|$, *cf.* [wiki bra-ket]. This notation owes its name to the 'bracket' notation $\langle\text{label 1}|\text{label 2}\rangle$ for inner products, consisting of a left part, $\langle\text{label 1}|$ called the 'bra,' and a right part, $|\text{label 2}\rangle$, called the 'ket.'

We will use this notation to name functionals and rank one operators built from vectors. That is, if $u, v \in H$, then $\langle u|$ denotes the linear functional $\langle u| : H \ni x \mapsto \langle u, x\rangle \in \mathbb{C}$ and $|u\rangle\langle v|$ the operator $|u\rangle\langle v| : H \ni x \mapsto \langle v, x\rangle u \in H$.

The vector $u(\theta,\phi)$ can be visualized as the point (θ,ϕ) on the unit sphere (*Bloch sphere*) in $\mathbb{R}^3$, that is, the vector

$$\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Density matrices are of the form

$$\rho(x,y,z) = \frac{I + x\sigma_x + y\sigma_y + z\sigma_z}{2}$$

with $x, y, z \in \mathbb{R}$, $x^2 + y^2 + z^2 \leq 1$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices.

Note that the density matrix associated with to a vector state $u(\theta,\phi)$ is simply

$$|u(\theta,\phi)\rangle\langle u(\theta,\phi)| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} = \rho\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

The state $\varphi_\rho : B(H) \to \mathbb{C}^2$ associated with to a density matrix ρ is the linear functional defined by

$$\varphi_\rho(X) = \mathrm{tr}(\rho X)$$

for $X \in B(H)$. If $\rho = |\psi\rangle\langle\psi|$ is pure state, that is, of the form $\rho = |\psi\rangle\langle\psi|$ for some vector $|\psi\rangle$, then this becomes

$$\varphi_{|\psi\rangle\langle\psi|} = \langle\psi, X\psi\rangle.$$

Observables (self-adjoint operators) are of the form

$$X = a|u\rangle\langle u| + b|u_\perp\rangle\langle u_\perp|,$$

with $a, b \in \mathbb{R}$, u a unit vector, $u_\perp$ orthogonal to u (unique up to a phase). In any experiment, X takes values a and b, with probabilities

$$P(X = a) = \varphi\big(|u\rangle\langle u|\big) \text{ and } P(X = b) = \varphi\big(|u_\perp\rangle\langle u_\perp|\big),$$

if φ is the state of the quantum system before the measurement.

After the experiment the state will be $|u\rangle\langle u|$, if the value a was observed, and $|u_\perp\rangle\langle u_\perp|$, if the value b was observed.

Suppose that

$$u(\theta, \phi) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

is the state vector labeled by the point

$$\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$

on the Bloch sphere, with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. Then we can take

$$u_\perp(\theta, \phi) = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix} = u(\pi - \theta, \phi + \pi)$$

for the vector orthogonal to $u(\theta, \phi)$. Note that $u_\perp(\theta, \phi)$ corresponds to the opposite point on the Bloch sphere.

Let us define

$$\begin{aligned} S(\theta,\phi) &= |u\rangle\langle u| - |u_\perp\rangle\langle u_\perp| \\ &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} \\ &= x\sigma_x + y\sigma_y + z\sigma_z \end{aligned}$$

where

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

is a point on the unit sphere in $\mathbb{R}^3$.

If we measure the observable $X = S(\theta,\phi)$ on a particle whose state is given by the state vector $u(\theta',\phi')$, we get

$$\begin{aligned} P(X=+1) &= \left|\langle u(\theta,\phi), u(\theta',\phi')\rangle\right|^2 = \frac{1+\cos\vartheta}{2} \\ P(X=-1) &= \frac{1-\cos\vartheta}{2} \end{aligned} \tag{1.2.1}$$

where ϑ is the angle between the points on the Bloch sphere that correspond to $u(\theta,\phi)$ and $u(\theta',\phi')$.

We can interprete the observable $S(\theta,\phi)$ as the measurement of the spin of an electron in the direction determined by θ and ϕ. The only two possible outcomes of this experiment are '$+1$' and '-1', which means that the spin points in the direction of the vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$

or that it points in the opposite direction, respectively.

To each observable X in a quantum probability space we can associate a classical probability space, with $\Omega = \sigma(X)$ and $P(\{\lambda\}) = \mathrm{tr}(\rho E_\lambda)$, if $X = \sum_{\lambda\in\sigma(X)} \lambda E_\lambda$ is the spectral decomposition of X.