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978-1-107-14790-4 — Collision Phenomena in Liquids and Solids
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1 Introduction

This introductory chapter overviews the fundamentals of collision phenomena in liquids and solids. It begins with the physical estimates in Section 1.1, which ascertain the conditions of the commonality of phenomena characteristic of liquid and solid collisions and the historical and modern reasons for deep interest in them. Before embarking on a discussion of the governing equations some basic dimensionless groups are introduced in Section 1.2. Then, the reader encounters the basic laws of mechanics of liquids and solids formulated as the mass and momentum balance equations in Section 1.3. The distinction between liquids and solids can stem from rheological constitutive equations, which are to be added to the basic laws. Two rheological models, of an inviscid and Newtonian viscous liquid, are introduced in Section 1.4, which transforms the basic laws to the Laplace equation for the kinematics of potential flows of inviscid fluids accompanied by the Bernoulli integral of the momentum balance, as well as to the Navier–Stokes equations describing general flows of viscous fluids, or in the limiting case, to the Stokes equations for the creeping flows dominated by viscosity. A special case of a strong short impact of solid onto any type of liquid reveals the potential impulsive motions introduced in Section 1.5. On the other hand, high-speed flows of low-viscosity liquids near a solid surface reveal traditional boundary layers, while near free liquid surfaces the other, less frequently discussed, boundary layers arise. Both types of the boundary layers and the corresponding equations are considered in Section 1.6. Geometric peculiarities of flows in thin liquid layers on solid surfaces allow for such simplifications as the quasi-one-dimensional and lubrication approximations discussed in Section 1.7. Special physical conditions exist at the moving contact line where liquid surface is in contact with both the underlying solid surface and the surrounding gas, which involves such issues as the Navier slip also covered in Section 1.7. The static configurations of sessile and pendant liquid drops, in particular their contact angles with solid surfaces, can be significantly affected by the surface texture and chemical composition – the group of questions elucidated in Section 1.8 and associated with wettability. Rheological transition from traditional liquids to solids is gradual and spans Newtonian viscous liquids, various non-Newtonian liquids including viscoelastic liquids, the elastic Hookean solids, elastic-viscoplastic materials and then, paradoxically (at very high impact velocities) the inviscid materials characterized by inertia only (Section 1.9). A short exposition of some basic instabilities encountered in collision phenomena is given in Section 1.10. Finally, in Section 1.11 the correct use of the energy balance approach in the modeling of some hydrodynamic problems is discussed.

1.1 History and Outlook

Collision phenomena are common, spectacular and frequent in real life. People were always fascinated with water drops impacting soil, stone, puddles or plants during rain. Watching rain generates thoughts, some physical, some philosophical, or both:

The rain to the wind said,
'You push and I'll pelt.'
They so smote the garden bed
That the flowers actually knelt,
And lay lodged–though not dead.
I know how the flowers felt.

"Lodged" by Robert Frost (1874–1963)[1]

Drops impacting onto a liquid layer are so attractive to the general public that they are regularly used in commercials aired on television, frequent on advertising billboards and shown on postcards. They motivated the famous poetic words of Edgerton and Killian (1954) in their book on ultra-high-speed photography: "In the land of splashes, what the scientist knows as Inertia and Surface Tension are the sculptors in liquids, and fashion from them delicate shapes none-the-less beautiful, because they are too ephemeral for any eye but that of the high-speed camera."

Drops impacting on liquid or solid surfaces can spread, or splash or even bounce back, as the detailed observations initiated in a series of brilliant works of Worthington in the late nineteenth century and summarized in his book Worthington (1908). To recognize the minute (actually, millisecond!) details of drop impact beyond those visible to poets, Worthington used high-speed photography, while the illumination was provided by a synchronized electric spark in air. The modern reincarnation of Worthington's approach is the use of charge-coupled devices (CCD cameras) and light-emitting diodes (LEDs) as light sources (Yarin 2006, Thoroddsen et al. 2008, Josserand and Thoroddsen 2016).

Drop spreading, splashing and bouncing imply an easy deformability characteristic of liquids, which are normally experienced as soft materials. However, folk wisdom expressed in the proverb "drop by drop wears away the stone" implies drop capabilities comparable to that of stones, for example, limestones located under leaky and dripping gutters. The characteristic time of water drop deformation during an impact, τ_{def}, is mostly determined by the competition of the inertia (the driving mechanism) and the surface tension (the restraining mechanism), and thus is of the order of $\tau_{def} \sim (\rho D^3/\sigma)^{1/2}$, where ρ and σ are the density and surface tension, and D is the volume-equivalent drop diameter. For water drops, with $\rho = 10^3$ kg/m^3, $\sigma = 0.0072$ kg/s^2 and $D \sim 0.001$ m, $\tau_{def} \sim 4$ ms, which indeed, requires a CCD camera for detailed observations. On the other hand, the impact time, τ_{imp}, is of the order of $\tau_{imp} \sim D/V_0$, where V_0 is the impact velocity. Therefore, in the cases where the impact velocity is high enough for the inequality $\tau_{imp} < \tau_{def}$ to hold, an impacting drop does not have enough

[1] Courtesy of Henry Holt and Company, LLC; The Random House Group, Penguin Random House, UK

time to deform and can initially behave as an extremely rigid (very stiff) solid. This determines the critical (lowest) limit of such solid-like behavior as $V_{0,crit} \approx (\sigma/\rho D)^{1/2}$, which is about 0.27 m/s. It should be emphasized that drops falling from a leaking gutter of a suburban home have velocities of the order of several meters per second, i.e. $V_0 > V_{0,crit}$, and thus exhibit an initially solid-like behavior. To evaluate the pressure they exert on an underlying surface, note that the information that the leading edge of a drop has impacted on an obstacle spreads with the speed of sound in water c, which is about 1497 m/s. During time Δt the mass of liquid, which is affected by the deceleration due to drop impact is thus $m = \rho c \Delta t S$, where S is the impact area. Accordingly, the momentum balance reads $mV_0 = F\Delta t$, with F being the force exerted on the underlying surface. Thus, the pressure p experienced by the underlying surface during the time $\tau_{el} \sim D/c \sim 10^{-6}s$ before rarefaction proceeds from the trailing side of the drop ($\tau_{el} \ll \tau_{def} < \tau_{imp}$), is $p = F/S = \rho c V_0$, with $V_0 > V_{0,crit}$. For $V_0 = 4$ m/s, this pressure is about $p \approx 60$ atm. For limestone, marble and granite the ultimate strength in compression can be as low as 20, 50 and 70 MPa (about 200, 500 and 700 atm), respectively, which means that a prolonged dripping can definitely wear them away and drop impacts, indeed, reveal some solid-like phenomena on the liquid side.

Cannon balls, bullets, projectiles and shaped-charge jets and their action on a target (a fortification or armor) attracted human attention not less intense than that devoted to rain, and especially their penetration capabilities were the focus of attention. In such cases one deals with sub-ordnance, ordnance and ultra-ordnance velocity ranges encompassing velocities from 25 to 3000 m/s (Backman and Goldsmith 1978). The field of terminal ballistics dealing with such questions was established by the classical works of Euler, Robins and Poncelet (Rosenberg and Dekel 2012), which were followed much later by the seminal works of Munroe (1900), Birkhoff et al. (1948) and Lavrentiev (1957). The early pioneers in the eighteenth and nineteenth centuries processed a wide variety of experimental data to establish the resistance experienced by cannon balls and bullets penetrating into solid targets, as well as the corresponding penetration depth. Only much later it was realized that in many cases solid–solid penetration reveals liquid-like properties of solids. For metals the yield stress Y and the ultimate strength σ_*, which is typically of the order of Y, are much less than the pressure exerted initially by a projectile, $p = \rho c V_0$ (exactly due to the same reason as for liquids), or at a later stage when it reduces to the level of $p = \rho V_0^2$ due to the rarefaction emanating from the rear edge. Indeed, taking for steel $\rho = 7.8 \times 10^3$ kg/m^3, $Y = 690$ MPa, $c \approx 5900$ m/s and V_0=1000 m/s, one finds the following ratios $Y/(\rho c V_0) = 0.015$ and $Y/(\rho V_0^2) = 0.088$. Similarly, for tungsten when $\rho = 19.25 \times 10^3$ kg/m^3, $Y = 550$ MPa and $c \approx 5220$ m/s, one finds for the collision velocity of V_0=1000 m/s, the following ratios $Y/(\rho c V_0) = 0.00547$ and $Y/(\rho V_0^2) = 0.0286$. Therefore, in cases of collision of steel and tungsten with armor, the pressure in both projectile and target far exceeds their plasticity limits, which means that metals will flow. Moreover, the above-mentioned low values of the Y/p ratios reveal that plastic resistance to flow will be relatively small, and the dominant forces will be inertial (the situation quite similar to that in flows of such "inviscid" liquids as water, especially after sufficiently fast drop impacts).

Neither the general public, nor the majority of the scientific community, realize that such spectacular phenomena as comet and asteroid collisions with planets, or projectile penetration into armor, can be "close relatives" of tiny drop impacts on the other end of the scale bar; however in fact, they are! The elucidation of this fact is the main motivation of the authors to write this book, since their personal research experiences spanned liquid–solid, liquid–liquid, solid–liquid and solid–solid impacts. Collision phenomena one encounters in real life, technology and nature span the entire spectrum from tiny drops to asteroids; to name a few:

- Ink-jet printing
- Spray cooling of hot surfaces
- Spray coating, spray painting
- Annealing, quenching of metal alloys
- Fire suppression
- Fuel injection
- Touchless cleaning with sprays
- Spray inhalation (impacts and deposition in the lungs)
- Encapsulation
- Domestic applications (e.g. hair spray)
- Near-net shape manufacturing
- Erosion of (steam) turbine blades
- Ice accretion on turbine components, power lines, aircraft
- Dilution of lubricating films due to fuel droplet impingement
- Spreading of plant diseases by rain
- Spore spreading by rain
- Criminal forensics
- Crop spraying
- Aeration of surface layers of lakes, seas and oceans
- Soil erosion
- Transport of granular materials
- Seaplane landing
- Shaped-charge jet penetration
- Ballistic penetration
- Military applications
- Explosion welding
- Solid material testing

Such a wide variety of fascinating and practically important situations typically involve a hidden common denominator dictated by "inviscid"-like flow and geometrical similarity of collision and impact phenomena. As the above-mentioned historical introduction shows, to a large part, the topics covered in this book have developed quite independently from one another in the sense that different communities were involved in the different collision phenomena: liquid–solid, liquid–liquid, solid–liquid and solid–solid. This book is an attempt to provide a unique vision of the underlying similarities existing in collision phenomena, which greatly facilitate their understanding and modeling and which are still not fully recognized due to the scatter of these phenomena among different disciplines.

The fact that each of these disciplines dealing with collision phenomena has undergone rapid development over the past years is indisputable. This can be attributed to a multitude of factors; however, there is no doubt that in the interest of improving numerous industrial processes (including those of modern high-tech industries), understanding the underlying physics, as opposed to relying on simple engineering correlations, is becoming a necessity and is increasingly being sought by industry. An in-depth understanding of various natural collision phenomena is also required to facilitate solid foundations of ecology, geology and other branches of science. It should be emphasized that joint consideration of fluid and solid mechanics including fracture mechanics is not uncommon for textbooks, as in the recent one of Barenblatt (2014), which shows that such an approach can be fruitful.

Collision phenomena are becoming recognized as one of the fundamental events on which an entire production or natural process may depend. This is most easily illustrated by the above-mentioned examples – both in engineering and in nature – in which impact and collision phenomena play a vital role.

Another factor contributing to the current interest in collision phenomena is undoubtedly the remarkable development in high-speed imaging over the past decade. This allows collision phenomena to be studied at unprecedented precision and resolution, revealing physics which were heretofore often only the subject of speculation or empirical modeling. Accordingly, new mathematical models of the phenomena can now be developed and validated to a much higher degree of certainty. Therefore, understanding and modeling of collision phenomena also form a challenging new domain in the fields of applied physics and mathematics, stimulating novel and classical experimental, theoretical and numerical approaches.

Whereas the book underlines similarities among different collision phenomena, there are some restrictions in scales. At very large length scales, for instance the collision of galaxies, or at very high velocity scales (hyper-ordnance or cosmic) phase transition, nuclear physics, gravity and relativity affect the collision phenomena, going beyond the scope of this book. Therefore, we can say that the book is restricted to mesoscales and ordnance and ultra-ordnance velocities, although attention is definitely paid to the effect of nano-texture on solid surfaces on drop impact, i.e. phenomena at nano-scales.

1.2 Dimensionless Groups

Dimensional analysis is a powerful tool for generalization of experimental data and uncovering hidden scalings and self-similarities in seemingly complicated hydrodynamic situations. The general ideas and multiple examples of the applications of dimensional analysis are discussed in several superb monographs, which an interested reader can easily find: Bridgman (1931), Barenblatt (1987, 2000), Sedov (1993) and Yarin (2012). Therefore, in the present section we briefly list the main dimensionless groups relevant in this book. The dimensionless groups governing drop impact onto a solid surface or a liquid layer are

$$\mathrm{We} = \frac{\rho D V_0^2}{\sigma}, \quad \mathrm{Re} = \frac{\rho D V_0}{\mu}, \quad \mathrm{Oh} = \frac{\mu}{(\rho\sigma D)^{1/2}} = \frac{\mathrm{We}^{1/2}}{\mathrm{Re}} \tag{1.1}$$

$$\mathrm{K} = \mathrm{We}\cdot\mathrm{Oh}^{-2/5}, \quad \mathrm{St} = \frac{M V_0}{6\pi\mu a^2}, \quad \mathrm{H} = \frac{h_0}{D}, \quad \mathrm{R} = \frac{\rho_g}{\rho}, \quad \mathrm{Vi} = \frac{\mu_g}{\mu} \tag{1.2}$$

where ρ, μ and σ denote liquid density, viscosity and surface tension, D and V_0 the drop diameter and impact velocity, h_0 thickness of the liquid film, ρ_g and μ_g are the surrounding gas density and viscosity. We, Re and Oh denote the Weber, Reynolds and Ohnesorge numbers, and H dimensionless film thickness; K is an important composite group. St is the Stokes number, where M is the mass of a spherical particle of radius $a = D/2$ impacting onto a thin viscous layer of viscosity μ at the wall. In addition, R and Vi denote the density and viscosity ratios. Also, gravity-related effects are characterized by the Bond number Bo = $\rho g D^2/\sigma$, i.e. the ratio of D^2 to the square of the capillary length

$$\lambda_c = \left(\frac{\sigma}{\rho g}\right)^{1/2} \tag{1.3}$$

(g being gravity acceleration), or by the Froude number

$$\mathrm{Fr} = \frac{V_0^2}{gD} = \frac{\mathrm{We}}{\mathrm{Bo}}. \tag{1.4}$$

Further dimensionless parameters characterizing roughness and wettability effects will be relevant to drop impact on solid dry surfaces, as well as the equilibrium contact angle. Among them the capillary number

$$\mathrm{Ca} = \frac{\mu V_0}{\sigma} = \frac{\mathrm{We}}{\mathrm{Re}}. \tag{1.5}$$

The capillary number is important if the dynamic contact angle influences significantly the considered flow. In this case the velocity of propagation of the contact line U is used in the expression (1.5) instead of V_0.

Non-spherical drop aspect ratio and the Strouhal number characterizing transient phenomena can also appear.

In relation to the discussion of the electrohydrodynamic aspects of drop impacts, the dimensionless charge relaxation time α, and the electric Bond number Bo_E naturally arise, with

$$\alpha = \frac{\tau_C V_0}{D}, \quad \mathrm{Bo}_E = \frac{D E_\infty^2}{\sigma} \tag{1.6}$$

where τ_C is the charge relaxation time, and E_∞ is the applied electric field strength, or alternatively, U_E/D, where voltage U_E is given.

Moreover, and this is very important for understanding of the organization of Parts I to V of the book, the dimensionless groups related to the rheological behavior of colliding materials can be introduced. Namely, the Deborah number De and the dimensionless stiffness S and plasticity P groups (or, alternatively, the modified Bingham number, Bn) can be introduced as

$$\mathrm{De} = \frac{\theta V_0}{D} \tag{1.7}$$

$$\mathrm{S} = \frac{E}{\rho V_0^2}, \;\; \mathrm{P \text{ or } Bn} = \frac{Y}{\rho V_0^2}, \tag{1.8}$$

where θ is the viscoelastic relaxation time, E is Young's modulus and Y is the yield stress.

In addition, at the early stages of impact or collision when the compressibility effects are important, instead of Eqs. (1.8), the dimensionless groups S, P or Bn should involve the speed of sound in solid material c, i.e.

$$\mathrm{S} = \frac{E}{\rho c V_0}, \;\; \mathrm{P \text{ or } Bn} = \frac{Y}{\rho c V_0}. \tag{1.9}$$

Also, the expressions for the groups P and Bn can involve the ultimate strength σ_* rather than the yield stress Y, as in Eqs. (1.8) and (1.9).

The dimensionless groups (1.1)–(1.7) will determine the discussion in Parts I–IV of the book. In particular, the effects of Re, We, R, Vi and especially K groups on drop deposition, splashing and bouncing will be introduced in the ordered manner. It should be emphasized that the density and viscosity ratios would only be important in very specific situations, not all that common. The dimensionless groups (1.8) and (1.9) are relevant for the discussion in Part V.

Note also, that the significant number of dimensionless groups listed above (even not including such dimensionless groups as the Stefan number (Ste) related to thermal effects in Section 4.2 in Chapter 4, or the Mach number (Ma) related to the gas compressibility in Section 4.8 of Chapter 4), does not allow one to strictly order the material according to only Re and We, but makes much more reasonable the present organization of material in the book, which ascends from the liquid-only to the solid-only phenomena, with the secondary details (and the corresponding dimensionless groups) discussed in the framework of this structure.

1.3 Mass and Momentum Balance Equations

Here we restrict ourselves to the incompressible case mostly relevant to problems related to flows of liquids associated with drop impact. The mass balance equation for an incompressible liquid reduces in hydrodynamics to the so-called continuity equation in the following invariant form

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{1.10}$$

This scalar equation is insufficient alone to describe fluid flows, since it incorporates two or three velocity components. Furthermore, not every arbitrary velocity vector will satisfy this equation and only those velocity fields which do not contradict Eq. (1.10) are kinematically admissible and thus can be realized, in principle. Those velocity fields which do not satisfy Eq. (1.10) are forbidden, since they lead to hole or fold formation in the flow field, i.e. to discontinuities, which are not permitted by the continuity equation (1.10).

The momentum balance equation in hydrodynamics is nothing but the second law of Newton for an infinitesimally small material element (Lamb 1959, Loitsyanskii 1966, Landau and Lifshitz 1987, Batchelor 2002)

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{a}, \tag{1.11}$$

where the incompressibility is assumed. The material time derivative is denoted as $\mathrm{D}(\bullet)/\mathrm{D}t$, ρ is the density, $\boldsymbol{\sigma}$ is the stress tensor (related to the surface forces) and $\boldsymbol{a}$ is the acceleration associated with a body force. If the body force is restricted to be the gravity force, then $\boldsymbol{a} = \boldsymbol{g}$, with $\boldsymbol{g}$ being acceleration due to gravity.

The fluid particle acceleration expressed by the material time derivative $\mathrm{D}\mathbf{v}/\mathrm{D}t$ can be split into the temporal and convective parts, namely

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}. \tag{1.12}$$

This expression shows that even if the velocity field is stationary and $\partial \mathbf{v}/\partial t = 0$, a material particle will still experience an acceleration when it is entrained by flow to a location with a different local velocity, which is expressed by the second term on the right-hand side of Eq. (1.12).

In mechanics of incompressible fluids the stress tensor $\boldsymbol{\sigma}$ is traditionally split into two parts: an isotropic one associated with pressure p, and an additional, deviatoric tensor $\boldsymbol{\tau}$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}, \tag{1.13}$$

where $\boldsymbol{I}$ denotes the unit tensor.

Substituting Eq. (1.13) into Eq. (1.11), and using Eq. (1.12), one arrives at the following form of the momentum balance equation

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \right] = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{a}, \tag{1.14}$$

which is known as the Cauchy momentum equation.

The mass and momentum balance Eqs. (1.10) and (1.11) form a system of fundamental equations required to describe fluid flow. However, these equations are insufficient, since a statement about material behavior is required to relate the stress tensor $\boldsymbol{\sigma}$ with flow kinematics. It should be emphasized that Eqs. (1.10) and (1.11) apply equally to such different continua as incompressible elastic solids and incompressible fluids. There is therefore a need to distinguish different types of material behavior, which requires an additional rheological constitutive equation, which relates to flow kinematics. Sometimes (but very infrequently, e.g. for polymeric liquids; see Doi and Edwards 1986, Bird et al. 1987) such an equation can be derived from a micromechanical model of material of a certain type using methods of statistical physics. Alternatively (and much more frequently; cf. Loitsyanskii 1966, Landau and Lifshitz 1987, Larson 1988, Batchelor

2002) such an equation is postulated phenomenologically to mimic and generalize certain experimental observations of material behavior in some simplifying limiting cases. In Section 1.9 a detailed account of the phenomenological approach to the formulation of rheological constitutive equations of rheologically complex liquids and solids is given, whereas in the following section the two most important and simplest cases of the inviscid and viscous Newtonian fluids are covered.

1.4 Inviscid and Viscous Newtonian Fluids: The Incompressible Euler and Navier–Stokes Equations

Historically the first phenomenological tensorial rheological constitutive equation was introduced by Euler. He assumed that the deviatoric stresses (already understood at that time as viscous stresses after Newton's experiments) are negligibly small and thus the stress tensor is always isotropic, as in hydrostatics, even though the fluid is in motion

$$\boldsymbol{\tau} = \mathbf{0}, \quad \boldsymbol{\sigma} = -p\boldsymbol{I}. \tag{1.15}$$

Bearing in mind Eq. (1.15), the momentum balance (1.14) reduces to

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right] = -\boldsymbol{\nabla}p + \rho\boldsymbol{a}, \tag{1.16}$$

which is known as the (incompressible) Euler equation.

It should be emphasized that this nonlinear equation for an inviscid fluid can be analytically integrated if the body forces are conservative, i.e. possess a potential (which is true, for example, for the gravity force), irrespective of the fluid being incompressible or compressible [albeit barotropic, i.e. $\rho = \rho(p)$]. The integral is called the Bernoulli equation. For incompressible potential flows $\mathbf{v} = \boldsymbol{\nabla}\phi$ (which is equivalent to irrotational flows with $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$), where ϕ is the hydrodynamic potential, the continuity equation (1.10) reduces to the Laplace equation for ϕ

$$\nabla^2\phi = 0, \tag{1.17}$$

whereas the Bernoulli integral reads

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{(\boldsymbol{\nabla}\phi)^2}{2} + gz = f(t), \tag{1.18}$$

with g being the magnitude of the gravity acceleration (i.e. $a = g$), z being the vertical coordinate and $f(t)$ being a function of time which can be established from the boundary conditions. In such cases the kinematics of any fluid mechanical problem is generated by the corresponding solutions of the Laplace equation (1.17), whereas the dynamics, i.e. the corresponding pressure, are immediately recovered from the algebraic Bernoulli equation (1.18).

This simplifying approach, known as potential flow theory or ideal fluid flows, still may be rather involved when complicated free surface configurations are present and their evolution must be established. Such situations may require numerical solutions or further simplifications discussed below (see Section 1.5). For the numerical simulations of drop impact onto liquid surfaces (Weiss and Yarin 1999) it is convenient to use the integral equivalent of the Laplace equation (Lamb 1959, Tikhonov and Samarskii 1990), which allows one to find the normal velocity component at the free surface $v_n = \partial\phi/\partial n$ using the knowledge of the distribution of the potential ϕ at the free surface. Since the tangential velocity components can be found by differentiation of the known distribution of ϕ over the free surface, the entire velocity vector at the surface can be found using the information on ϕ only at the free surface. This forms the foundation of the Boundary Integral Method (or discretized, numerical equivalent, the Boundary Element Method, BEM). It is emphasized that the effect of the boundary associated with a solid wall underneath the liquid layer can be accounted for using the method of images (Weiss and Yarin 1999). Then, only the free liquid surface is left to be tackled. The time marching required to update the positions of the individual fluid elements at the free surface involves the kinematic condition there

$$\frac{\mathrm{D}\boldsymbol{r}}{\mathrm{D}t} = \boldsymbol{\nabla}\phi, \tag{1.19}$$

with $\boldsymbol{r}$ being the position vector, and the equation required to update the potential distribution at the free surface, which follows from the Bernoulli equation (1.18)

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{(\boldsymbol{\nabla}\phi)^2}{2} - \frac{\sigma\kappa}{\rho} - gz. \tag{1.20}$$

In this equation the pressure at the free surface is obtained invoking the Young–Laplace equation $p = \sigma\kappa$, with κ being the mean curvature of the free surface, σ the surface tension and $f(t) = 0$, when a droplet is already connected to a liquid layer, which extends to infinity. Note that for potential flows, the material time derivative

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + (\boldsymbol{\nabla}\phi)^2 \tag{1.21}$$

where x, y and z are the Cartesian coordinates, and u, v and w are the corresponding velocity components.

Potential flow, which is identically irrotational, is rooted in the simplified rheological constitutive equation (1.15). Indeed, an initially potential/irrotational flow stays a potential/irrotational flow at any time under the conditions of Kelvin's circulation theorem [zero viscosity, as in Eq. (1.15), conservative body forces, and fluid is barotropic] (Kochin et al. 1964, Batchelor 2002). Intuitively it is approximately valid for low-viscosity liquids (e.g. those like water), albeit, as was established much later in 1904 by Prandtl, only at some distance away from the solid boundaries (see Section 1.6 in the present chapter). For sufficiently viscous fluids and/or in cases where the flow development sufficiently close to a wall is studied, Eq. (1.15) is insufficient and hydrodynamics according to potential flow theory collapses. An alternative rheological constitutive equation is needed. This is the Newton–Stokes constitutive equation, which assumes a linear dependence between the deviatoric stress tensor $\boldsymbol{\tau}$ and the rate-of-strain tensor $\boldsymbol{D} = (\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^{\mathrm{T}})/2$, with $\boldsymbol{\nabla}\mathbf{v}$ being the tensor gradient of velocity. Namely, for