Cambridge University Press
978-1-107-14789-8 — Exact Solutions in Three-Dimensional Gravity
Alberto A. García-Díaz
Excerpt
More Information

1

Introduction

Given the large number of exact solutions that exist today in (2 + 1) Einstein gravity the purpose of the present book is to present a complete and concise list of exact solutions with emphasis on their physical and geometrical properties from the beginnings of the field in 1963 to the present, to be useful for the audience of experts and young researchers. Emphasis is given to solutions to the Einstein equations in the presence of matter and fields, for instance, point particle solutions, perfect fluids, cosmological spacetimes, dilatons, inflatons, and stringy solutions. The second part of this book deals with solutions to vacuum topologically massive gravity with a cosmological constant, as there exist three big families of spacetimes: the inhomogeneous Bianchi class of solutions, the Kundt spacetimes and the Cotton type N wave fields.

To avoid unnecessary typing, the cosmological constant is denoted by Λ, AdS spacetime stands for an asymptotically anti-de Sitter spacetime with $\Lambda < 0$, dS spacetime stands for an asymptotically de Sitter spacetime with $\Lambda > 0$, $3D$ stands for three dimensions, while $(2+1)D$ spacetime denotes $(2+1)$-dimensional spacetime, PF stands for perfect fluid, and ρ or μ denotes the fluid energy density. Occasionally we use SL for spacelike, TL for timelike, and ST to denote spacetime. On the other hand, when publications by various authors are cited, an abbreviation of their family names, including their first capital initials, are given; for instance, EEqs. and EM mean, respectively, Einstein equations and Einstein–Maxwell, MTW stands for Misner, Thorne, and Wheeler; FRW reads Friedmann–Robertson–Walker, and BTZ denotes Bañados–Teitelboim–Zanelli.

1.1 Main Features of (2 + 1) Gravity

In the early work by Giddings, Abbott, and Kuchař (GAK) (Giddings et al., 1984) it is stated that "the lowest dimension in which the Einstein Theory makes sense is $n = 3$." Consequently, bearing in mind the self-contained nature of this book, the main features of $(2+1)$-dimensional gravity are presented following the GAK pattern and essentially maintaining their wording.

1.1.1 Field Equations and Curvature Tensors

Einstein's theory of relativity, as a theory of a gravitational spacetime, can be based on two postulates which are independent of the spacetime dimensions; these postulates demand that the field equations take the form of the Einstein equations:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \kappa\, T_{\mu\nu} - \Lambda\, g_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy–momentum tensor (which, by virtue of the Bianchi identity, fulfills the energy conservation condition ${T^{\mu}}_{\nu;\mu} = 0$), Λ is a cosmological constant, and κ is a coupling constant. And second, the spacetime geometry is determined by the Riemann curvature tensor ${R^{\alpha}}_{\beta\gamma\delta}$.

The Riemann tensor in three dimensions possesses six algebraically independent components: as many as the number of independent components of the Ricci tensor; therefore, the Riemann tensor is completely determined by the Ricci tensor and the scalar curvature, namely:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= g_{\alpha\gamma}\, R_{\beta\delta} - g_{\alpha\delta}\, R_{\beta\gamma} - g_{\beta\gamma}\, R_{\alpha\delta} + g_{\beta\delta}\, R_{\alpha\gamma} \\ &\quad - \frac{1}{2}(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma})R. \end{aligned} \tag{1.2}$$

There is no room for the Weyl conformal tensor, which is thus zero.

Due to the Einstein equations, the Riemann tensor can be expressed in terms of the Einstein tensor or, in turn, through the energy momentum tensor $T_{\alpha\gamma}$ as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \kappa[g_{\alpha\gamma}\, T_{\beta\delta} - g_{\alpha\delta}\, T_{\beta\gamma}] + \kappa[g_{\beta\delta}\, T_{\alpha\gamma} - g_{\beta\gamma}\, T_{\alpha\delta}] \\ &\quad - \kappa(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma})T. \end{aligned} \tag{1.3}$$

In three dimensions, the coupling constant κ is measured in units of 1/mass, and therefore defines a natural mass unit.

1.1.2 Matter Distribution Locally Curves the Spacetime

Since, in a $3D$ spacetime, the Riemann curvature tensor is expressible solely through the energy–momentum tensor (1.3), thus, in an empty spacetime, where $T_{\alpha\gamma} = 0$, the spacetime is locally flat:

$$R_{\alpha\beta\gamma\delta} = 0.$$

Therefore, the flat spacetime is the field solution to the vacuum Einstein equations $G_{\mu\nu} = 0 \rightarrow R_{\mu\nu} = 0 = R$, ($T_{\mu\nu} = 0$ and $\Lambda = 0$). Staruszkiewicz (1963), in his pioneering article, stressed this fact by writing: "three-dimensional gravitation theory is a theory without a field of gravitation; where no matter is present, space is flat. Curvature can arise only if matter or energy are present."

1.1.3 Point Particles Produce Global Effects on the Spacetime

The first publication on gravity in $(2+1)$ dimensions by Staruszkiewicz (1963) was devoted to the description of static solutions determined by point sources. Point particles move along geodesics. In the points where the particles are located there arise conical defects (conical singularities) that can be felt at infinity; the total mass in the spacetime is proportional to the deficit angle at infinity. Because the angle deficit cannot increase by 2π, the mass is bounded from above. See Chapter 2 for details.

1.1.4 Newtonian Limits

This section has to be subdivided into three subsections: first, to recall the content of the Newtonian theory; second, to reveal the existence of the Newtonian limit in the standard $(3+1)$, or $(1+(n-1))$, gravity via the weak gravitational field treatment; and finally, to show that the slow motion limit of the $(2+1)$ gravity occurs without acceleration.

Newtonian Theory of Gravity

The Newtonian theory of gravity is based on the Newtonian potential ϕ fulfilling the Poisson equation with matter density ρ,

$$\nabla^2 \phi = 4\pi\, G\, \rho, \tag{1.4}$$

which generates the Newtonian field causing the accelerated motion of test particles in it:

$$\frac{d^2\, x^i}{dt^2} = -\delta^{i\,j} \partial_j\, \phi. \tag{1.5}$$

G stands for the Newton constant of gravitation.

Weak Gravitational Theory in n Dimensions

In this paragraph, starting from the n-dimensional Einstein equations for weak gravitational fields (expansion of the metric components), the equation obeyed by the weak fields, and related to the energy density, is derived. The limit of the geodesic equation of motion for test particles in the case of slow velocities and weak gravitational fields is derived and its consequences analyzed.

GAK Linearized Approach

Following GAK, the linearized Einstein equations in the Hilbert–de Donder gauge reduce to the inhomogeneous wave equation

$$\Box\, h_{\alpha\beta} = -2\kappa \left[\tau_{\alpha\beta} - \frac{\tau}{n-2}\eta_{\alpha\beta}\right]. \tag{1.6}$$

For small perturbations,

$$h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta},\, (\eta_{\alpha\beta}) = \mathrm{diag}\,(-1,1,1,1). \tag{1.7}$$

In weak fields, the linearized stresses τ_{ij} (spatial components) are negligible in comparison to the mass density $\tau_{00} = \rho$, and assuming additionally the quasi-staticity of the field and sources, one arrives from (1.6) at

$$\nabla^2\, h_{00} = -2\kappa\, \frac{n-3}{n-2}\, \tau_{00}, \tag{1.8}$$

or, identifying $h_{00} = -2\,\phi$, one gets

$$\nabla^2\,\phi = \kappa\, \frac{n-3}{n-2}\,\rho. \tag{1.9}$$

The geodesic equation

$$\frac{dx^\alpha}{ds^2} + \Gamma^\alpha{}_{\mu\nu}\, \frac{dx^\mu}{ds}\, \frac{dx^\nu}{ds} = 0, \tag{1.10}$$

for slow motion in the linearized limit, becomes

$$\frac{dx^i}{dt^2} - \frac{1}{2}\,\delta^{ij}\partial_j\, h_{00} = 0 \to \frac{dx^i}{dt^2} = -\,\delta^{ij}\partial_j\,\phi. \tag{1.11}$$

Carlip Linearized Approach

In Chapter 1 of Carlip (1998), it is stated that "general relativity in 2+1 dimensions has a Newtonian limit in which there is no force between static point masses." The starting point to establish this assessment is the approximation (1.7), and it continues with the n-dimensional field equations given in the harmonic gauge (which can always be chosen) such that:

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\,\partial_\nu\,\bar{h}_{\alpha\beta} + \mathcal{O}(h^2) = \kappa T_{\alpha\beta},$$
$$\eta^{\mu\nu}\partial_\mu\,\bar{h}_{\nu\beta} = 0, \tag{1.12}$$

where

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\,\eta^{\mu\nu}h_{\mu\nu} \to h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{n-2}\eta_{\alpha\beta}\,\eta^{\mu\nu}\bar{h}_{\mu\nu}. \tag{1.13}$$

The Newtonian limit is obtained:

by setting $T_{00} = \rho$, where ρ is the mass density;
by equating to zero all other components of the stress–energy tensor;
by ignoring time derivatives;

then, identifying

$$\bar{h}_{00} = -4\phi, \tag{1.14}$$

the linearized equations (1.12) reduce to

$$\nabla^2\phi = \frac{\kappa}{2}\,\rho. \tag{1.15}$$

In this limit, the geodesic equation (1.10) reduces to

$$\frac{d^2x^i}{dt^2} - \frac{1}{2}\,\partial_i\,h_{00} = 0 \tag{1.16}$$

and, taking into account (1.13) and (1.14), becomes

$$\frac{d^2x^i}{dt^2} = -2\frac{n-3}{n-2}\,\delta^{ij}\partial_j\,\phi. \tag{1.17}$$

Discrepancies

Comparing these results reported by GAK, Giddings et al. (1984), and Carlip (1998), one notices that the numerical coefficients in the GAK equations for the limits of Newton and particle motion equations correspond to those in the limit of the geodesic motion and to the Newton equation respectively of Carlip (1998).

The Outcome of the Dilemma

A detailed derivation of the linearized Einstein theory is done in $3+1$ dimensions by Tonnelat (1959), Chapter 12, §$1, 2, 3$, which can be extended to $(1+(n-1))$ dimensions, practically without any changes. In this manner, using the de Donder conditions, $\sigma^\beta = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\,g^{\alpha\,\beta}) = 0$ – which allow for the existence of (isothermal) harmonic coordinates and the use of the subclass of quasi-Lorentzian coordinates in the linearization problem of the Einstein equations – one gets the Newton limit in the form (1.15). Moreover, the limit of the geodesic equation for slow motion is also established by Tonnelat (1959), which in terms of the "bar" quantities, in the terminology of MTW, Misner et al. (1973, Chapter 18), gives (1.16), and consequently, in term of the Newtonian potential, it reduces to (1.17).

Weak Gravitational Theory in $(2+1)$ Dimensions

Having at hand the linearized expressions of the Einstein equations and of the geodesic equations, in any dimension n, by means of the equations (1.15) and (1.17), correspondingly, one easily recognizes that in $(2+1)$ gravity the Newtonian limit holds in the two spatial dimensions, but the Newtonian acceleration equation fails to be true; the geodesic slow motion occurs without acceleration, $\frac{d^2x^i}{dt^2} = 0$.

1.1.5 No Geodesic Deviation for Dust

It is apparent that the geodesic deviation for neighboring moving particles has to vanish. Consider a congruence of geodesics with tangent vectors u^α, and let the separation vectors from one geodesic to another be V^μ; then the geodesic deviation equation is

$$\nabla_u \nabla_u V^\alpha = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma V^\delta. \tag{1.18}$$

Assume now that this congruence is modeled by a tube of dust with energy–momentum tensor

$$T^{\alpha\beta} = \rho u^\alpha u^\beta. \tag{1.19}$$

Substituting this tensor into the expression of the Riemann tensor (1.3) and contracting it with u, one gets

$$R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma = 0, \tag{1.20}$$

which in turn implies

$$\nabla_u \nabla_u V^\alpha = 0. \tag{1.21}$$

Therefore, in a $(2+1)$ spacetime the world lines of dust do not deviate. In particular, the trajectories of point particles do not approximate to one another; in other words, there is no acceleration between them. This final observation is another effect which is present in $(2+1)$ gravity. Summarizing, in this theory there is *no action at distance:* in a $3D$ spacetime, gravitational effects do not propagate outside the matter content; test particles outside the matter region move along geodesics without experiencing acceleration and geodesic deviation.

1.1.6 No Dynamic Degrees of Freedom

In more than three dimensions, the Weyl tensor encodes the information about the Riemann curvature not caused by matter. Since in $3D$ spacetime the Weyl tensor vanishes – that is, there is no room for it – then, because curvature is only produced by matter, the gravitational field has no dynamic degrees of freedom. Another way to arrive at this absence of degrees of freedom is through a counting argument appealing to the canonical geometrodynamics, as was done in $3D$ by Giddings et al. (1984) and by Carlip (1998) in nD. Roughly speaking, in the canonical geometrodynamics, the spacetime is foliated by means of a one-parameter family of spacelike hypersurfaces, $x^\alpha = x^\alpha(x^a, t)$; thus one can define an intrinsic metric

$$g_{ab} = g_{\alpha\beta} X^\alpha_a X^\beta_b, \; X^\alpha_a := \frac{\partial x^\alpha}{\partial x^a}, \; a = 1, \ldots, n-1, \; \alpha = 1, \ldots, n-1, t,$$

and a field of unit normals U^α, to these hypersurfaces, together with the extrinsic curvatures, $K_{ab} = -U_{\alpha;\beta} X^\alpha_a X^\beta_b$. The Einstein equations are then decomposed with respect to the normal and tangential directions to the hypersurfaces; then we introduce the lapse function N, the shift vector N^a, the intrinsic metric g_{ab} and the momentum conjugate to the metric p^{ab}, defined by means of the extrinsic metric, which are elevated to the category of canonical variables. Einstein equations are then expressed in terms of all of them, with constraints: the projections $G_{\alpha\beta} U^\alpha U^\beta$, (1), $G_{\alpha\beta} U^\alpha X^\beta_a$, $(n-1)$, the evolution eq. for p: $\dot{p}^{ab} \sim G_{ab} X^\alpha_a X^\beta_b$, $n(n-1)/2$, and the evolution eq. for g: $\dot{g}_{ab}$, $n(n-1)/2$.

In n dimensions, the intrinsic metric possesses $n(n-1)/2$ components, and the conjugate momentum also has $n(n-1)/2$ components, and together the number of their components is $n(n-1)$. On the other hand, one can fix n of them by choosing n coordinates, and additionally n by the constraints; consequently, the number of degree of freedom in the canonical data is: $n(n-1) - 2n = n(n-3)$. Hence, in three dimensions there is no freedom in the prescription of the initial data in the initial hypersurface.

As a consequence of this lack of degrees of freedom, there are *no gravitational waves in 3D flat spacetime*; in the terminology of Gott and Alpert (1984), there are no gravity waves in flatland; no gravitons.

1.1.7 Black Holes in (2 + 1) Gravity

So there are no black holes in asymptotically flat spacetime; the asymptotically flat space is flat everywhere, and as such it does not allow for any solution different to the one corresponding to the Minkowskian $3D$ metric. However, let us consider a different situation from the flat asymptotic: for asymptotically anti-de Sitter $(2+1)$ spacetime there exists a black hole solution found by Bañados, Teitelboim, and Zanelli: the BTZ black hole, Bañados et al. (1992). Since that original discovery, various classes of black hole solutions, in the presence of fields and matter, have been reported in the literature.

1.1.8 Gravity in the Presence of Other Fields and Matter

$3D$ gravity in the presence of other fields is worthy of deep study; many achievements have been made since studies began. The present text is simply designed to show the existence of big families of exact solutions in three dimensions and their parallelism, if any, with those classes of the standard $4D$ gravity. For instance, one may attempt a bridge between $3D$ conformally flat metrics and nD conformally flat metrics for incompressible perfect fluids, Friedmann–Robertson–Walker cosmology, and FRW dilaton–inflaton theory, among others.

1.2 Algebraic Classification

In $4D$ gravity, to characterize adequately the gravitational field, there exist two main classifications: the Petrov classification of the Weyl conformal tensor – Petrov types of gravitational fields – and the Plebański–Pirani classification of the tensor of matter or of the traceless Ricci tensor. In (2+1) gravity the situation simplifies considerably: one is dealing with symmetric tensors for the matter from one side, and, to characterize the conformal properties of the metric, the Cotton tensor from the other.

1.2.1 Classification of the Cotton–York Tensor

The role of the conformal tensor in $(2+1)$ gravity is played by the Cotton tensor, Cotton (1899); see Stephani et al. (2003) for a more recent reference. In

$3D$ geometry, the conformal property of the space is guaranteed by the vanishing of the conformal Cotton tensor. This tensor is defined by means of the covariant derivatives of the Ricci tensor and of the scalar curvature according to

$$C^{\alpha\beta} = C^{\beta\alpha} = \eta^{\mu\nu(\alpha} \left(R^{\beta)}{}_{\mu} - \frac{1}{4} R\, \delta^{\beta)}{}_{\mu} \right)_{;\nu}, \tag{1.22}$$

where the symmetry has been introduced explicitly. Notice that the Cotton tensor is traceless:

$$C^{\alpha}{}_{\alpha} = 0. \tag{1.23}$$

To classify the Cotton tensor with respect to its eigenvalues, one has to solve a generalized eigenvalue problem:

$$\left(C^{\alpha\beta} - \lambda\, g^{\alpha\beta}\right) V_{\beta} = 0\,, \quad C^{[\alpha\beta]} = 0\,, \quad C^{\alpha\beta}\, g_{\alpha\beta} = 0\,. \tag{1.24}$$

By lowering one index, one can reformulate this task as an ordinary eigenvalue problem for the matrix $C_{\alpha}{}^{\beta}$. However, in that case, the symmetry $C^{\alpha\beta} = C^{\beta\alpha}$ is no longer present:

$$\left(C_{\alpha}{}^{\beta} - \lambda\, \delta^{\beta}_{\alpha}\right) V_{\beta} = 0\,, \quad C_{\alpha}{}^{\alpha} = 0\,. \tag{1.25}$$

Accordingly, the matrix $C_{\alpha}{}^{\beta}$ is no longer symmetric and the roots of the characteristic polynomial

$$\det \left(C_{\alpha}{}^{\beta} - \lambda\, \delta^{\beta}_{\alpha}\right) = 0 \tag{1.26}$$

may be complex. This point seems to have been overlooked by Barrow et al. (1986).

I will present a classification of $C_{\alpha}{}^{\beta}$ along the lines of Garcia–Hehl–Heineke–Macías (GHHM), García et al. (2004), where the components are referred to an orthonormal basis:

$$g = g_{\alpha\beta} dx^{\alpha}\, dx^{\beta} = \eta_{ab}\, \Theta^{a}\, \Theta^{b},\ (\eta_{ab}) = \mathrm{diag}(-1, 1, 1). \tag{1.27}$$

The trace-free condition $(1.25)_2$ reads explicitly

$$C_1{}^1 + C_2{}^2 + C_3{}^3 = 0\,. \tag{1.28}$$

Accordingly, we can eliminate $C_3{}^3$, e.g., from $(1.25)_1$. Then the secular determinant reads

$$\det \begin{vmatrix} C_1{}^1 - \lambda & C_1{}^2 & C_1{}^3 \\ -C_1{}^2 & C_2{}^2 - \lambda & C_2{}^3 \\ -C_1{}^3 & C_2{}^3 & -C_1{}^2 - C_2{}^2 - \lambda \end{vmatrix} = 0\,, \tag{1.29}$$

with the five matrix elements $C_1{}^2, C_1{}^2, C_1{}^3, C_2{}^2, C_2{}^3$. The equation to determine the eigenvalues λ amounts explicitly to

$$\lambda^3 + b\,\lambda + c = 0\,, \tag{1.30}$$

where

$$b := -(C_1{}^1)^2 - C_1{}^1 C_2{}^2 - (C_2{}^2)^2 + (C_1{}^2)^2 + (C_1{}^3)^2 - (C_2{}^3)^2\,, \tag{1.31}$$

$$c := \left[({C_1}^1)^2{C_2}^2 + {C_1}^1({C_2}^2)^2 + {C_1}^1({C_1}^2)^2 + {C_1}^1({C_2}^3)^2 + ({C_1}^2)^2{C_2}^2 + 2\,{C_1}^2{C_1}^3{C_2}^3 - ({C_1}^3)^2{C_2}^2\right] . \tag{1.32}$$

The roots of (1.29) are given by

$$\lambda_1 = A\,, \quad \lambda_2 = -\frac{A}{2} + i\frac{\sqrt{3}}{2}B\,, \quad \lambda_3 = -\frac{A}{2} - i\frac{\sqrt{3}}{2}B\,, \tag{1.33}$$

with

$$A := \frac{D^2 - 12b}{6D}\,, \quad B := \frac{D^2 + 12b}{6D}\,, \quad D := \left(-108c + 12\sqrt{12b^3 + 81c^2}\right)^{1/3} .$$

A cubic polynomial with real coefficients has at least one real root and the complex roots have to be complex conjugates. The Petrov types, Jordan normal forms and Segré notations of the Cotton tensor read:

Table 1.2.1 *Algebraic classification of the Cotton tensor*

“Petrov” type	Jordan form	Segré notation	eigenvalues relation
I	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}$	[111]	$\lambda_1 \neq \lambda_2,\ \lambda_3 = -\lambda_1 - \lambda_2$
D	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	[(11)1]	$\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1$
II	$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	[21]	$\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1$
N	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	[(21)]	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
III	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	[3]	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
O	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$		

This parallels exactly the Petrov classification of the Weyl tensor in four dimensions in Stephani et al. (2003). This comes about since the Weyl tensor in $4D$ is equivalent to a (complex) 3×3 trace-free matrix, as ${C_\alpha}^\beta$ in $3D$; for a similar classification of $C_{\alpha\beta}$, see Hall and Capocci (1999). A detailed derivation of the Cotton tensor in any dimension together with an account of its properties is presented here in Chapter 20; see also García et al. (2004).

1.2.2 Classification of the Energy–Momentum Tensor

The standard classification of the energy–momentum tensor T_{ab} takes advantage of its symmetry property, $T_{ab} = T_{ba}$, where the Latin letters denote the indices with respect to the orthonormal basis (1.27). For that reason the eigenvectors are found by solving the matrix equation

$$T_{ab}\,V^b = \lambda\,\eta_{ab}\,V^b, \rightarrow (T_{ab} - \lambda\,\eta_{ab})\,V^b = 0. \tag{1.34}$$

Searching the values of λ that cancel the determinant of the matrix

$$\begin{bmatrix} T_{11}+\lambda & T_{12} & T_{13} \\ T_{12} & T_{22}-\lambda & T_{23} \\ T_{13} & T_{23} & T_{33}-\lambda \end{bmatrix}, \tag{1.35}$$

namely the roots of the eigenvalue polynomial

$$\begin{aligned}
&\lambda^3 + c_2\,\lambda^2 + c_1\,\lambda + c_0 = 0,\\
&c_0 := T_{11}, T_{22}\,T_{33} - T_{13}{}^2\,T_{22} + 2\,T_{12}\,T_{23}\,T_{13} - T_{12}{}^2\,T_{33} - T_{11}\,T_{23}{}^2,\\
&c_1 := -T_{11}\,T_{22} - T_{11}\,T_{33} + T_{12}{}^2 + T_{22}\,T_{33} + T_{13}{}^2 - T_{23}{}^2,\\
&c_2 := T_{11} - T_{22} - T_{33},
\end{aligned} \tag{1.36}$$

which allows for three roots, with its possible degenerations,

$$\begin{aligned}
\lambda_1 &= -\frac{c_2}{3} + \frac{1}{6}\sqrt[3]{Rd} - 6\,F\,{}^{-3}\!\sqrt{Rd},\\
\lambda_{2,3} &= -\frac{c_2}{3} - \frac{1}{12}\sqrt[3]{Rd} + 3\,F\,{}^{-3}\!\sqrt{Rd} \pm \frac{1}{12}\,i\sqrt{3}\left(\sqrt[3]{Rd} + 36\,F\,{}^{-3}\!\sqrt{Rd}\right),\\
Rd &:= 36\,c_1\,c_2 - 108\,c_0 - 8\,c_2{}^3 + 12\,D,\\
D &:= \sqrt{12\,c_1{}^3 - 3\,c_1{}^2 c_2{}^2 - 54\,c_1\,c_2\,c_0 + 81\,c_0{}^2 + 12\,c_0\,c_2{}^3},\\
F &= \frac{1}{3}\,c_1 - \frac{1}{9}c_2{}^2,
\end{aligned} \tag{1.37}$$

one would be able to determine the eigenvectors corresponding to each root.

The nomenclature used for eigenvectors and algebraic types of tensors is borrowed from Plebański (1964): timelike, spacelike, null, and complex vectors are denoted respectively by **T**, **S**, **N**, and **Z**. For algebraic types are used the symbols: $\{\lambda_1 T, \lambda_2 S_2, \lambda_3 S_3\} \equiv \{T, S, S\}$, meaning that the first real eigenvalue λ_1 gives rise to a timelike eigenvector **T**, the second real eigenvalue λ_2 is associated with a spacelike eigenvector $\mathbf{S}_2$, and finally the third real eigenvalue λ_3 is related to a spacelike eigenvector $\mathbf{S}_3$; for the sake of simplicity I use the symbols $\{T, S, S\}$. It is clear that $\{N, N, S\}$ stands for the algebraic type allowing for two different real eigenvalues giving rise to two null eigenvectors while the third real root is associated with a spacelike eigenvector. When there are single and double real eigenvalues giving rise correspondingly to timelike and spacelike eigenvectors, the algebraic type is denoted by $\{T, 2\,S\}$; consequently, for a triple real eigenvalue, if that were the case, the types could be $\{3T\}$, $\{3N\}$, or $\{3S\}$. For a complex eigenvalue λ_Z, in general, the related eigenvectors are complex and are denoted by **Z** and $\bar{\mathbf{Z}}$ for its complex conjugate; the possible types are $\{T, Z, \bar{Z}\}$, $\{N, Z, \bar{Z}\}$, or $\{S, Z, \bar{Z}\}$.