

## Numerical Linear Algebra

This self-contained introduction to Numerical Linear Algebra provides a comprehensive, yet concise, overview of the subject. It includes standard material such as direct methods for solving linear systems and least-squares problems, error, stability and conditioning, basic iterative methods and the calculation of eigenvalues. Later chapters cover more advanced material, such as Krylov subspace methods, multigrid methods, domain decomposition methods, multipole expansions, hierarchical matrices and compressed sensing.

The book provides rigorous mathematical proofs throughout, and gives algorithms in general-purpose language-independent form. Requiring only a solid knowledge in linear algebra and basic analysis, this book will be useful for applied mathematicians, engineers, computer scientists and all those interested in efficiently solving linear problems.

HOLGER WENDLAND holds the Chair of Applied and Numerical Analysis at the University of Bayreuth. He works in the area of Numerical Analysis and is the author of two other books, *Scattered Data Approximation* (Cambridge, 2005) and *Numerische Mathematik* (Springer 2004, with Robert Schaback).

Cambridge Texts in Applied Mathematics

All titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit [www.cambridge.org/mathematics](http://www.cambridge.org/mathematics).

*Nonlinear Dispersive Waves*

MARK J. ABLOWITZ

*Flow, Deformation and Fracture*

G. I. BARENBLATT

*Hydrodynamic Instabilities*

FRANÇOIS CHARRU

*The Mathematics of Signal Processing*

STEVEN B. DAMELIN & WILLARD MILLER, JR

*Introduction to Magnetohydrodynamics (2nd Edition)*

P. A. DAVIDSON

*An Introduction to Stochastic Dynamics*

JINQIAO DUAN

*Singularities: Formation, Structure and Propagation*

J. EGGERS & M. A. FONTELOS

*A Physical Introduction to Suspension Dynamics*

ÉLISABETH GUAZZELLI & JEFFREY F. MORRIS

*Discrete Systems and Integrability*

J. HIETARINTA, N. JOSHI & F. W. NIJHOFF

*Iterative Methods in Combinatorial Optimization*

LAP CHI LAU, R. RAVI & MOHIT SINGH

*An Introduction to Polynomial and Semi-Algebraic Optimization*

JEAN BERNARD LASSERRE

*An Introduction to Computational Stochastic PDEs*

GABRIEL J. LORD, CATHERINE E. POWELL & TONY SHARDLOW

# Numerical Linear Algebra

## An Introduction

HOLGER WENDLAND  
*Universität Bayreuth, Germany*



CAMBRIDGE  
UNIVERSITY PRESS

CAMBRIDGE  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom  
One Liberty Plaza, 20th Floor, New York, NY 10006, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India  
79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107147133](http://www.cambridge.org/9781107147133)

DOI: 10.1017/9781316544938

© Holger Wendland 2018

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2018

Printed in the United Kingdom by Clays, St Ives plc

*A catalogue record for this publication is available from the British Library.*

ISBN 978-1-107-14713-3 Hardback

ISBN 978-1-316-60117-4 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet Web sites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

## Contents

	<i>Preface</i>	<i>page ix</i>
	<b>PART ONE PRELIMINARIES</b>	<b>1</b>
<b>1</b>	<b>Introduction</b>	<b>3</b>
	1.1 Examples Leading to Linear Systems	5
	1.2 Notation	10
	1.3 Landau Symbols and Computational Cost	13
	1.4 Facts from Linear Algebra	17
	1.5 Singular Value Decomposition	24
	1.6 Pseudo-inverse	26
	Exercises	29
<b>2</b>	<b>Error, Stability and Conditioning</b>	<b>30</b>
	2.1 Floating Point Arithmetic	30
	2.2 Norms for Vectors and Matrices	32
	2.3 Conditioning	46
	2.4 Stability	54
	Exercises	55
	<b>PART TWO BASIC METHODS</b>	<b>57</b>
<b>3</b>	<b>Direct Methods for Solving Linear Systems</b>	<b>59</b>
	3.1 Back Substitution	59
	3.2 Gaussian Elimination	61
	3.3 LU Factorisation	65
	3.4 Pivoting	71
	3.5 Cholesky Factorisation	76

3.6	QR Factorisation	78
3.7	Schur Factorisation	84
3.8	Solving Least-Squares Problems	87
	Exercises	100
<b>4</b>	<b>Iterative Methods for Solving Linear Systems</b>	101
4.1	Introduction	101
4.2	Banach's Fixed Point Theorem	102
4.3	The Jacobi and Gauss–Seidel Iterations	106
4.4	Relaxation	116
4.5	Symmetric Methods	125
	Exercises	130
<b>5</b>	<b>Calculation of Eigenvalues</b>	132
5.1	Basic Localisation Techniques	133
5.2	The Power Method	141
5.3	Inverse Iteration by von Wielandt and Rayleigh	143
5.4	The Jacobi Method	153
5.5	Householder Reduction to Hessenberg Form	159
5.6	The QR Algorithm	162
5.7	Computing the Singular Value Decomposition	171
	Exercises	180
	<b>PART THREE ADVANCED METHODS</b>	181
<b>6</b>	<b>Methods for Large Sparse Systems</b>	183
6.1	The Conjugate Gradient Method	183
6.2	GMRES and MINRES	203
6.3	Biorthogonalisation Methods	226
6.4	Multigrid	244
	Exercises	258
<b>7</b>	<b>Methods for Large Dense Systems</b>	260
7.1	Multipole Methods	261
7.2	Hierarchical Matrices	282
7.3	Domain Decomposition Methods	307
	Exercises	327
<b>8</b>	<b>Preconditioning</b>	329
8.1	Scaling and Preconditioners Based on Splitting	331
8.2	Incomplete Splittings	338
8.3	Polynomial and Approximate Inverse Preconditioners	346

<i>Contents</i>		vii
8.4	Preconditioning Krylov Subspace Methods	357
	Exercises	368
<b>9</b>	<b>Compressed Sensing</b>	370
9.1	Sparse Solutions	370
9.2	Basis Pursuit and Null Space Property	372
9.3	Restricted Isometry Property	378
9.4	Numerical Algorithms	384
	Exercises	393
	<i>Bibliography</i>	395
	<i>Index</i>	403

## Preface

Numerical Linear Algebra (NLA) is a subarea of Applied Mathematics. It is mainly concerned with the development, implementation and analysis of numerical algorithms for solving linear problems. In general, such linear problems arise when discretising a continuous problem by restricting it to a finite-dimensional subspace of the original solution space. Hence, the development and analysis of numerical algorithms is almost always problem-dependent. The more is known about the underlying problem, the better a suitable algorithm can be developed.

Nonetheless, many of the so-derived methods are more general in the sense that they can be applied to larger classes of problems than initially intended. One of the challenges in Mathematics is deciding how to describe the necessary assumptions, under which a certain method works, in the most general way. In the context of NLA, this means finding for each method the most general description of matrices to which the method can be applied. It also means extracting the most general methods from the vast number of available algorithms. Particularly for users with new problems this is crucial, as it allows them to apply and test well-established algorithms first, before starting to develop new methods or to extend existing ones.

In this book, I have attempted to use this *matrix-driven* approach rather than the *problem-driven* one. Naturally, the selection of the material is biased by my own point of view. Also, a book on NLA without any examples would be rather dire, so there are typical examples and applications included to illustrate the methods, but I have tried to restrict myself to simple examples, which do not require much previous knowledge on specific problems and discretisation techniques.

During the past years, I have given courses on Numerical Linear Algebra at advanced BSc and early MSc level at the University of Sussex (UK), the University of Oxford (UK) and the University of Bayreuth (Germany). I have also



given courses on Numerical Analysis which covered parts of the NLA material in Oxford, Göttingen (Germany) and Bayreuth.

This book on Numerical Linear Algebra is based on these courses and the material of these courses. It covers the standard material, as well as more recent and more specific techniques, which are usually not found in standard textbooks on NLA. Examples include the multigrid method, the domain decomposition method, multipole expansions, hierarchical matrices and applications to compressed or compressive sensing. The material on each of these topics fills entire books so that I can obviously present only a selection. However, this selection should allow the readers to grasp the underlying ideas of each topic and enable them to understand current research in these areas.

Each chapter of this book contains a small number of theoretical exercises. However, to really understand NLA one has to implement the algorithms by oneself and test them on some of the matrices from the examples. Hence, the most important exercises intrinsic to each chapter are to implement and test the proposed algorithms.

All algorithms are stated in a clean pseudo-code; no programming language is preferred. This, I hope, allows readers to use the programming language of their choice and hence yields the greatest flexibility.

Finally, NLA is obviously closely related to Linear Algebra. However, this is not a book on Linear Algebra, and I expect readers to have a solid knowledge of Linear Algebra. Though I will review some of the material, particularly to introduce the notation, terms like linear space, linear mapping, determinant etc. should be well-known.