

## Index of Categories and Functors

Categories are printed in typewriter font

### Categories

BDLat, bounded distributive lattices, **586**  
 BDSLat, bounded distributive  
 join-semilattices, **99**  
 BiFr, bi-frames, **588**  
 BiLocSp, bi-localic spaces, **313**  
 BoolAlg, Boolean algebras, **19**  
 BoolSp, Boolean spaces, **16**  
 CReg, completely regular spaces, **289**  
 Comp, compact spaces, **289**  
 Fr<sup>op</sup>, opposite category of Fr, **299**  
 Fr, frames, **588**  
 HeytAlg, Heyting algebras, **278**  
 ISprings, category of indexed springs, **483**  
 InvFr, inverse frames, **588**  
 InvLocSp, inverse localic spaces, **313**  
 LocSp, localic spaces, **313**  
 PoSets, posets = partially ordered sets, **580**  
 Priestley, Priestley spaces, **33**  
 QoSets, quasi-ordered sets, **580**  
 RedRings, reduced rings, **439**  
 Rings, rings (commutative, unital), **67**  
 Sets, sets, **55**  
 Spec, spectral spaces, **11**  
 Springs, category of springs, **483**  
 T<sub>0</sub>Sob, sober T<sub>0</sub>-spaces, **384**  
 T<sub>0</sub>Top, T<sub>0</sub>-spaces, **138**  
 ToSets, totally ordered sets, **580**  
 Top, topological spaces, **11**  
 U, category of spaces with indeterminates,  
**473**  
 semiSpec, semi-spectral spaces, **99**

### Functors

$\lambda$ : Rings  $\rightarrow$  BDLat, reticulation functor,  
**431**  
 $\mathcal{O}$ : Top  $\rightarrow$  BDLat, **588**  
 $\mathcal{O}$ : Top  $\rightarrow$  Fr, **588**  
 $\mathcal{K}$ : Spec  $\rightarrow$  BDLat, **11**  
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 Q: Spec  $\rightarrow$  PoSets, **408**  
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## Symbol Index

The first entries contain symbols with no or ambiguous alphabetical value, sorted in order of appearance. After these, the symbols are ordered alphabetically.

**Order, see Appendix: The Poset Zoo, 579**

**Symbols without Alphabetic Value, in Order of Appearance**

- $\rightsquigarrow$ , specialization, 3
- $\rightsquigarrow_\tau$ , specialization (in topology  $\tau$ ), 3
- $X_{\text{con}}^\tau$ , constructible (= patch) topology of  $X$ , 16
- $\overline{S}^{\text{con}}$ , constructible closure of  $S$ , 16
- $f_{\text{con}}$ , map induced by  $f$  on patch spaces, 21
- $X_{\text{inv}}$ , inverse topology on  $X$ , 23
- $\overline{C}^{\text{inv}}$ , inverse closure of  $C$ , 23
- $f_{\text{inv}}$ , map induced by  $f$  on inverse spaces, 25
- $\llbracket \widehat{T} = k \rrbracket$ , set of maps in  $2^S$  with value  $k$  on  $T \subseteq S$ , 54
- $\llbracket \widehat{s} = k \rrbracket$ , set of maps in  $2^S$  with value  $k$  at  $s \in S$ , 54
- $X^{\rightsquigarrow}$ , graph of specialization of  $X$ , 59
- $X \times_S Y$ , fiber product (= pull-back) of  $X$  and  $Y$  over  $S$ , 61
- $X \times_S^{\rightsquigarrow} Y$ , specialization fiber product of  $X, Y$  over  $S$ , 63
- $X_1 \oplus X_2$ , topological sum of two spaces, 65
- $\bigoplus_{i \in I} X_i$ , topological sum (= coproduct) of  $X_i$ , 65
- $\Delta$ , symmetric difference in Boolean algebra, 71
- $A$ -isomorphism between morphisms in a category, 76
- $A$ -morphism between morphisms in a category, 76
- $\neg = \neg_A$ , complementation map in Boolean algebra  $A$ , 95
- 0-dimensional space, 122
- $[\cdot]_E$ , equivalence class for  $E$ , 134
- $\equiv_r$ , lattice filter congruence, 159
- $\equiv_l$ , lattice ideal congruence, 159

- $[\cdot]_i, [\cdot]_f$ , congruence classes in lattices, 159
- $X/R$ , spectral quotient of  $X$  modulo  $R$ , 170
- $X/T_0R$ ,  $T_0$ -quotient of  $X$  modulo  $R$ , 170
- $X/\text{set } E$ , 170
- $R^{\text{sat}}$ , saturation of relation  $R$ , 174
- $\leq_R$ , quotient order on  $X/\text{set } E$ , 176
- $\ll$ , way below (for subsets of poset), 207
- $\cdot \rightarrow \cdot$ , implication (Heyting algebra), 269
- $\sim a$ , pseudo-complement of  $a$ , 269
- $\langle S \rangle_L$ , sub-frame generated by  $S \subseteq L$ , 301
- $\coprod_{i \in I} X_i$ , spectral coproduct of  $X_i$ , 329
- $Y^{\text{clcon}}$ , set of closed and constructible points in spectral space  $Y$ , 397
- $\mathfrak{a} \triangleleft A$ ,  $\mathfrak{a}$  ideal of ring  $A$ , 421
- $A_S$ , ring of fractions with denominators in  $S$ , 435
- $\mathfrak{a}_S$ , ideal in ring of fractions  $A_S$  generated by canonical image of ideal  $\mathfrak{a}$ , 435
- $A_p$ , localization of  $A$  at prime ideal  $p$ , 436
- $\mathfrak{a}_p$ , ideal generated in localization  $A_p$  by canonical image of ideal  $\mathfrak{a} \triangleleft A$ , 436
- $b : c$ , quotient of ideals  $b, c$  in a ring, 439
- $\leq_\alpha$ , order determined by  $\alpha \in \text{Sper}(A)$  on field  $\text{qf}(A/\text{supp}(\alpha))$ , 490
- $\neg\Delta = \{\neg\delta \mid \delta \in \Delta\}$ , 542
- $\models$ , 542
- $X^{\leq}$ , graph of order relation  $\leq$  on  $X$ , 579
- $P^{\text{max}}$ , set of maximal elements of poset  $(P, \leq)$ , 580
- $P^{\text{min}}$ , set of minimal elements of poset  $(P, \leq)$ , 580
- $P_{\text{inv}}$ , inverse of poset  $P$ , 580
- $X^{\text{max}} = (X, \rightsquigarrow_\tau)^{\text{max}}$ , set of closed points of the topological space  $(X, \tau)$ , 581
- $X^{\text{min}} = (X, \rightsquigarrow_\tau)^{\text{min}}$ , set of generic points of the topological space  $(X, \tau)$ , 581

$\leq_{\text{inv}}$ , inverse of order relation  $\leq$ , 580  
 $\rightsquigarrow$ , proper specialization, 580  
 $\bar{Q} \leq p$ ,  $\forall q \in Q : q \leq p$ , 581  
 $\perp = \perp_P$ , smallest element of poset  $P$ , 581  
 $\top = \top_P$ , largest element in poset  $P$ , 581  
 $p \leq Q$ ,  $\forall q \in Q : p \leq q$ , 581  
 $p^\downarrow$ , principal down-set of  $p$ , 581  
 $p^\downarrow$ , 581  
 $p^\uparrow$ , principal up-set of  $p$ , 581  
 $p^\uparrow$ , 581  
 $Q^\downarrow$ , down-set generated by  $Q$ , 582  
 $Q^\uparrow$ , up-set generated by  $Q$ , 582  
 $\langle p \rangle$ , graph component of vertex  $p$  in a graph, 582  
 $[\cdot, \cdot]$ , closed interval = convex hull of two points, 583  
 $\vee Q = \text{sup}(Q)$ , supremum of  $Q$ , 585  
 $\wedge Q = \text{inf}(Q)$ , infimum of  $Q$ , 585  
 $\vee$ , join operation on lattice, 585  
 $\vee$ -semilattice = join-semilattice, 585  
 $\vee_{\text{inv}}$ , join operation in inverse of a  $\wedge$ -semilattice, 585  
 $\wedge$ , meet operation on lattice, 585  
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 $\wedge$ -irreducible = meet-irreducible, 586

**Alphabetic Symbol List**

$\mathbb{1}$ , one-element space, 8  
**2**, Boolean algebra with two elements, 15  
**2**, Sierpiński space, 8  
**3**, =  $\mathfrak{r}$  for  $n = 3, 9$   
 $\mathcal{A}(E; L)$ , closed elements (of completion  $E$  of  $L$ ), 303  
 $A_{\text{red}}$ , reduced ring associated with ring  $A$ , 439  
 $A^\times$ , group of units in ring  $A$ , 421  
 $\mathcal{A}(X)$ , set of closed sets of  $X$ , 2  
 $\text{ba}(L)$ , Boolean envelope of  $L$ , 93  
 $\text{ba}_L$ , embedding  $L \rightarrow \text{ba}(L)$ , 93  
 $\text{ba}(\varphi)$ , extension of lattice morphism  $\varphi$  to Boolean envelope, 94  
 $\beta_X : X \rightarrow \beta X$ , Stone–Čech compactification of the completely regular space  $X$ , 123  
 $\text{CB}(X)$ , Cantor–Bendixson rank of space  $X$ , 114  
 $\text{CB}_X(x) = \text{CB}(x)$ , Cantor–Bendixson rank of  $x \in X$ , 114

$\chi$ , characteristic functions map  $\mathfrak{P}(X) \rightarrow 2^X$ , 55  
 $\text{Clop}(X)$ , set of clopen subsets of  $X$ , 14  
 $\text{Cong}(L)$ , set of congruences of lattice  $L$ , 76  
 $\text{conv}(Q) = \text{conv}_P(Q)$ , convex hull of  $Q$  in  $P$ , 583  
 $\text{con}_X$ , identity  $X_{\text{con}} \rightarrow X$ , 21  
 $\text{Coz}(X)$ , lattice of cozero sets in topological space  $X$ , 287  
 $C(X, \mathbb{R})$ , ring of continuous functions on topological space  $X$  with values in  $\mathbb{R}$ , 287  
 $D(\cdot)$ , basic opens of Zariski topology, 67  
 $D(\cdot) = D_L^F(\cdot)$ , basic opens of  $\text{PrimF}(L)$ , 89  
 $D(\cdot), D_L^I(\cdot)$ , basic opens of  $\text{PrimI}(L)$ , 90  
 $D(\delta)$ , subbasic open sets of the space of  $\Delta$ -types, 544  
 $\delta X$ , set of non-isolated points of  $X$ , 114  
 $\delta^\alpha X, \delta^\infty X$ , iterations of  $\delta X$ , 114  
 $\Delta(X)$ , diagonal of  $X$ , 59  
 $\Delta_X$ , diagonal  $X \rightarrow X \times X$ , 59  
 $\Delta_A$ , positive quantifier-free  $\mathcal{L}(A)$ -sentences, 555  
 $\text{diag}(A)$ , diagram of structure  $A$ , 555  
 $\text{diag}^+(A)$ , positive diagram of structure  $A$ , 562  
 $\mathcal{E}(S)$ , set of equivalence relations on  $S$ , 75  
 $\mathcal{E}(X)$ , canonical extension of  $X$ , 304  
 $\mathbb{F}_2$ , field with two elements, 71  
 $\bar{f}(a)$ , principal filter generated by  $a$ , 83  
 $f_{\text{inv}}$ , map  $f$  between posets considered as a map between the inverse posets, 580  
 $\text{Fml}(\mathcal{L})$ , formulas of language  $\mathcal{L}$ , 542  
 $\Gamma(f)$ , graph of map  $f$ , 59  
 $\Gamma^*$ , finite words, 240  
 $\Gamma^{\leq n}$ , words of length at most  $n$ , 240  
 $\Gamma^{< n}$ , words of length  $< n$ , 240  
 $\Gamma_X$ , canonical map  $X \rightarrow \Gamma(X)$ , 199  
 $\Gamma(X)$ , set of connected components of  $X$ , 199  
 $\text{Gen}(A)$ , set of generalizations of elements of  $A$ , 3  
 $\mathcal{H}_{\mathcal{L}(A)}$ , Harrison topology on  $\mathcal{L}(A)$ , 500  
 $\text{Hom}_{\text{BDLat}}(L, \mathbf{2})$ , set of bounded distributive lattice homomorphisms of  $L$  to  $\mathbf{2}$ , 79  
 $H_{\mathbb{R}}^{\geq 0}(a_1, \dots, a_r)$ , basic opens of real spectrum, 72  
 $\text{ht}(\mathfrak{a})$ , height of ideal  $\mathfrak{a}$  in a ring, 452  
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 $\mathcal{I}^{\text{rad}}(A)$ , set of radical ideals of  $A$ , 421  
 $\mathcal{I}_{\text{fin}}^{\text{rad}}(A)$ , radical ideals generated by finite subsets of ring  $A$ , 422  
 $\mathcal{I}(R)$ , set of ideals of ring  $R$ , 74



- $I(P)$ , set of ideals of poset  $P$ , 588
- $i \triangleleft P$ , ideal in a poset, 588
- $i(a)$ , principal ideal generated by  $a$  in a lattice, 83
- $\inf(Q) = \bigwedge Q$ , infimum of  $Q$ , 585
- $\text{IntAlg}(L)$ , interval algebra of  $L$ , 98
- $\text{int}_X(Y)$ , interior of  $Y$  in space  $X$ , 121
- $\iota_p$ , localizing homomorphism at  $p$ , 436
- $\iota_S$ , canonical homomorphism to a ring of fractions, 435
- $k(\alpha)$ , real closure of field  $\kappa(\alpha)$ , 515
- $\kappa(\alpha)$ , ordered field  $\text{qf}(A/\text{supp}(\alpha), \leq_\alpha)$ , for  $\alpha \in \text{Sper}(A)$ , 490
- $\kappa(v)$ , valued field determined by valuation  $v$  of a ring, 569
- $\text{Kdim}(A)$ , Krull dimension of ring  $A$ , 114
- $\text{Kdim}(X)$ , Krull dimension of space  $X$ , 113
- $\text{Ker}^F$ , transfer map  
 $\text{Hom}_{\text{BDLat}}(L, \mathbf{2}) \rightarrow \text{PrimF}(L)$ , 89
- $\text{Ker}^I$ , transfer map  
 $\text{Hom}_{\text{BDLat}}(L, \mathbf{2}) \rightarrow \text{PrimI}(L)$ , 89
- $K(X)$ , set of compact elements in a poset  $X$ , 207
- $\overset{\circ}{\mathcal{K}}(X)$ , set of quasi-compact opens of  $X$ , 3
- $\overset{\circ}{\mathcal{K}}(\tau)$ , set of quasi-compact opens (in topology  $\tau$ ), 3
- $\overset{\circ}{\mathcal{K}}(f)$ , restriction of  $\mathfrak{F}(f)$  to  $\overset{\circ}{\mathcal{K}}(\cdot)$ , 11
- $\mathcal{K}(X)$ , set of constructible sets of  $X$  (clopens of  $X_{\text{con}}$ ), 16
- $\mathcal{K}(f)$ , map  $\mathcal{K}(Y) \rightarrow \mathcal{K}(X)$  induced by  $f: X \rightarrow Y$ , 21
- $\overline{\mathcal{K}}(X)$ , set of closed constructible sets of  $X$ , 16
- $\overline{\mathcal{K}}(f)$ , map  $\overline{\mathcal{K}}(Y) \rightarrow \overline{\mathcal{K}}(X)$  induced by  $f: X \rightarrow Y$ , 21
- $\overline{\mathcal{K}}(f)$ , map  $\overline{\mathcal{K}}(Y) \rightarrow \overline{\mathcal{K}}(X)$  induced by  $f: x \rightarrow Y$ , 25
- $\mathcal{L}(A)$ , denotes either  $\text{Qmod}(A)$ ,  $\text{Preord}(A)$ , or  $\text{Satpre}(A)$ , 500
- $\lambda_A$ , reticulation map of ring  $A$ , 429
- $\lambda_i$ , homomorphism  $L \rightarrow \mathbf{2}$  defined by prime filter  $\mathfrak{f}$ , 89
- $\lambda_i$ , homomorphism  $L \rightarrow \mathbf{2}$  defined by prime ideal  $i$ , 90
- $\Lambda$ , empty word, 240
- $\overset{\circ}{\Lambda}_X$ , Stone representation, 85
- $\overline{\Lambda}_X$ , Stone representation, 85
- $\text{Lb}(Q)$ , set of lower bounds of  $Q$ , 581
- $\mathcal{L}(C)$ , language  $\mathcal{L}$  extended by new constants, 542
- $L/\mathfrak{f}$ , factor lattice modulo filter  $\mathfrak{f} \subseteq L$ , 159
- $L/i$ , factor lattice modulo ideal  $i \subseteq L$ , 159
- $\ell\Delta$ , closure of set  $\Delta$  of sentences under conjunction and disjunction, 542
- $\ell(s)$ , length of word  $s$ , 240
- $\lim X_i$ , projective limit of  $X_i$ , 63
- $\overline{\text{LocCl}}(X)$ , set of locally closed points of  $X$ , 135
- $\mathcal{L}_{\text{or}}$ , language for ordered rings, 519
- $(M, \alpha)$ , expansion of structure  $A$  determined by homomorphism  $\alpha$ , 562
- $M \cdot A = (M)$ , ideal generated by subset  $M$  in ring  $A$ , 421
- $(M) = (M)_A = M \cdot A$ , ideal generated by subset  $M$  in ring  $A$ , 421
- $\mu(M)$ , multiplicative set generated by  $M$ , 421
- $\mu_s(M)$ , saturated multiplicative set generated by  $M$ , 421
- $N_*$ , inclusion of the image of nucleus  $N$  in the frame, 302
- $N^*$ , nucleus  $N$  with restricted codomain, 302
- $\mathcal{N}(G)$ , set of normal subgroups of group  $G$ , 75
- $\text{nil}(A)$ , nilradical in ring  $A$ , 421
- $\mathbb{N} = \{1, 2, \dots\}$ , natural numbers, 583
- $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ , non-negative integers, 583
- $N(O)$ , open regularization of  $O \in \mathcal{O}(X)$ , 129
- $\mathbf{n}$ , set  $\{0, 1, \dots, n-1\}$ , naturally ordered, 583
- $n$ , spectral space with elements  $0, 1, \dots, n-1$  in natural order, 9
- $N_{Y, X}$ , nucleus associated with localic subspace  $Y$  of  $X$ , 319
- $\mathcal{O}(E; L)$ , open elements (of completion  $E$  of  $L$ ), 303
- $\mathcal{O}(f)$ , restriction of  $\mathfrak{F}(f)$  to  $\mathcal{O}(\cdot)$ , 11
- $\omega = \mathbb{N}_0$ , smallest infinite ordinal, 584
- $\overline{N}(A)$ , closed regularization of  $A \in \mathcal{A}(X)$ , 130
- $\mathcal{O}(X)$ , set of open subsets of  $X$ , 2
- $\mathcal{O}_x(X)$ , filter of open neighborhoods of  $x \in X$ , 171
- $\mathfrak{F}_{\text{fin}}(I)$ , set of finite subsets of  $I$ , 63
- $\mathfrak{F}(f)$ , power set dual of  $f$ , 11
- $\pi_\alpha$ , canonical map  $A \rightarrow \kappa(\alpha)$  for  $\alpha \in \text{Sper}(A)$ , 490
- $\pi_a$ , canonical homomorphism from a ring to the factor ring modulo  $a$ , 435
- $\pi_{\mathfrak{f}}: L \rightarrow L/\mathfrak{f}$ , canonical quotient homomorphism, 159
- $\pi_i: L \rightarrow L/i$ , canonical quotient homomorphism, 159
- $\text{Preord}(A)$ , set of preorders of ring  $A$ , 499

- $\text{PrimF}(L)$ , set of prime filters of bounded distributive lattice  $L$ , 89  
 $\text{PrimF}(\varphi)$ , spectral dual of lattice morphism  $\varphi$ , 91  
 $\text{PrimI}(L)$ , set of prime ideals of bounded distributive lattice  $L$ , 89  
 $\text{PrimI}(\varphi)$ , spectral dual of lattice morphism  $\varphi$ , 91  
 $\mathbb{P}$ , set of prime numbers, 70  
 $\text{pt}_{\text{bi}}(Y)$ , set of bi-localic points of  $Y$ , 315  
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 $\text{pt}(Y)$ , set of localic points of  $Y$ , 315  
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 $qR, T_0$ , canonical map  $X \rightarrow X/T_0 R$  to  $T_0$ -quotient, 170  
 $qR : X \rightarrow X/R$ , canonical spectral quotient, 170  
 $q : X \rightarrow X/E, T_0$ -reflection of  $X$ , 171  
 $R^{-1}$ , inversion of relation  $R$ , 176  
 RCF = real closed field, 518  
 $RC(X)$ , set of regular closed sets in topological space  $X$ , 130  
 $\text{Rk}^\alpha(X)$ , elements in a poset with rank  $\geq \alpha$ , 116  
 $\text{Rk}(X)$ , rank of a poset or of a spectral space, 116  
 $\text{rk}(x) = \text{rk}_{(X, \leq)}(x)$ , rank of point  $x$  in poset  $(X, \leq)$ , 116  
 $RO(X)$ , set of regular open sets in space  $X$ , 130  
 $r = r_X$ , retraction of normal spectral space  $X$  onto  $X^{\max}$ , 283  
 $\mathbf{R}_X$ , localic coreflection map, 404  
 $\text{Satpre}(A)$ , set of saturated preorders of ring  $A$ , 499  
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 $S_{\text{loc}}(Y)$ , set of localic subspaces of  $Y$ , 315  
 $S_n(T)$ , space of  $n$ -types of theory  $T$ , 548  
 $\mathbf{Sob}_X$ , sobrification map of topological space  $X$ , 384  
 $\mathbf{Sob}(X)$ , sobrification of space  $X$ , 384  
 $\mathbf{Sob}(f)$ , sobrification of continuous map  $f$ , 385  
 $\text{Spec}(L)$ , spectrum of bounded distributive lattice  $L$ , 81  
 $\text{Spec}_p(A)$ ,  $p$ -adic spectrum of ring  $A$ , 572  
 $\text{Spec}(\varphi)$ , spectral dual of lattice homomorphism  $\varphi$ , 81  
 $\text{Spec}(\varphi)$ , map between Zariski spectra induced by a ring homomorphism  $\varphi$ , 70, 434  
 $\text{Spec}_{\text{re}}(\varphi)$ , map between real prime ideals induced by a ring homomorphism  $\varphi$ , 507  
 $\text{Spec}_{\text{re}}(R)$ , space of real prime ideals of  $R$ , 505  
 $\text{Spec}(R)$ , Zariski spectrum (or prime spectrum) of ring  $R$ , 67  
 $\text{Sper}(\varphi)$ , real spectral dual of ring morphism  $\varphi$ , 73  
 $\text{Sper}(R)$ , real spectrum of ring  $R$ , 72, 491  
 $\text{Spez}(A)$ , set of specializations of elements of  $A$ , 3  
 $\text{Spv}(A)$ , valuation spectrum of ring  $A$ , 569  
 $\sqrt{(M)}$ , radical ideal generated by  $M$ , 421  
 $S^*$ , one-point compactification of discrete space  $S$ , 44  
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 $\mathcal{U}(G)$ , set of subgroups of group  $G$ , 75

*Symbol Index*

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- |   |  |
|---|--|
| $\text{Up}(X)$ , family of up-sets of $(X, \leq)$ , 75                  | $V(\delta)$ , subbasic closed sets for the space of $\Delta$ -types, 544 |
| $V(\cdot)$ , basic closed sets of Zariski topology, 69                  | $Z(X)$ , lattice of zero sets in topological space $X$ , 287             |
| $V(\cdot) = V_L^F(\cdot)$ , basic closed sets of $\text{PrimF}(L)$ , 89 |  |
| $V(\cdot), V_L^I(\cdot)$ , basic closed sets of $\text{PrimI}(L)$ , 90  |  |

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