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Max Dickmann , Niels Schwartz , Marcus Tressl
Frontmatter
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Spectral Spaces

Spectral spaces are a class of topological spaces. They are a tool linking algebraic structures, in a very wide sense, with geometry. They were invented to give a functional representation of Boolean algebras and distributive lattices and subsequently gained great prominence as a consequence of Grothendieck's invention of schemes.

There are more than 1000 research articles about spectral spaces, but this is the first monograph. It provides an introduction to the subject and is a unified treatment of results scattered across the literature, filling in gaps and showing the connections between different results. The book includes new research going beyond the existing literature, answering questions that naturally arise from the comprehensive approach. The authors serve graduates by starting gently with the basics. For experts, they lead them to the frontiers of current research, making this book a valuable reference source.

MAX DICKMANN has been a researcher at the CNRS, France, since 1974; Directeur de Recherche since 1988; emeritus since 2007. His research interests include the applications of spectral spaces to real algebraic geometry, quadratic forms, and related topics.

NIELS SCHWARTZ is Professor of Mathematics at the Universität Passau, retired since 2016. Many of his publications are concerned with, or use, spectral spaces in essential ways. In particular, he has used spectral spaces to introduce the notion of real closed rings, an important topic in real algebra and geometry.

MARCUS TRESSL is a mathematician working in the School of Mathematics at the University of Manchester. His research interests include model theory, ordered algebraic structures, ring theory, differential algebra, and non-Hausdorff topology.

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MAX DICKMANN
Centre National de la Recherche Scientifique (CNRS), Paris, France

NIELS SCHWARTZ
Universität Passau, Germany

MARCUS TRESSL
University of Manchester, UK

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

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Contents

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Preface

Spectral spaces constitute a class of topological spaces used in various branches of mathematics. They were introduced in the 1930s by M. H. Stone and have been used extensively ever since. There was a marked growth of interest following A. Grothendieck's revolution of algebraic geometry. It was realized that spectral spaces can be associated with many mathematical structures. Numerous publications are devoted to various properties of spectral spaces and to a growing number of diverse applications. The area is extremely active and is growing at a fast pace.

With this book we provide the first comprehensive and coherent treatment of the basic *topological* theory of spectral spaces. It is possible to study spectral spaces largely with algebraic tools, namely using bounded distributive lattices, or, in more abstract form, using category theory, model theory, or topos theory. However, our focus is clearly on the topology, which provides geometric tools and intuition for applications that, *a priori*, do not have geometric meaning. Also, in our experience, the topological techniques are very flexible towards possible extensions of techniques and results to wider classes of spaces, where a corresponding algebraic framework does not exist.

We start with a careful analysis of the definition of spectral spaces, describe fundamental structural features, and discuss elementary properties. Numerous examples, counterexamples, and constructions, listed in an index of examples, show how one can work with spectral spaces in concrete situations or illustrate results. We exhibit methods illustrating how spectral spaces can be associated with different classes of structures and describe some of the most important applications.

It was our original intention to assemble basic material about spectral spaces in one place to make it more easily accessible. Collecting the material and preparing a coherent presentation proved to be more laborious than anticipated: the terminology and notation differ from publication to publication. The Zariski spectrum of commutative unital rings is undoubtedly the most widely used construction of spectral spaces. Therefore, many results on spectral spaces are found in publications about rings and are expressed in the corresponding language. These needed to be translated into topological language to make them compatible with our intentions and presentation. There are numerous points, some small, some more substantial, about basic topological properties of spectral spaces that had not been considered before, but have to be studied in a comprehensive treatment. So, to round off the picture, we had to fill various gaps in the existing literature. The construction of new spaces from a collection of given spaces is an important topic to which we added new facets. The category of spectral spaces (i.e., spectral spaces with spectral maps as morphisms) is naturally related to several other categories – for example, the categories of topological spaces and of partially ordered sets. The precise relationship with these other categories had previously been studied only in a fragmentary way. In our book we do this systematically, which requires the development of suitable tools.

Taking all of this into account, after starting off writing a book in the style of a graduate text, our project has turned, to a considerable extent, into a research monograph. We think that our text is suitable for graduate students who study spectral spaces and for researchers who find the subject relevant and need a solid basis for their own work. This may be in algebra, algebraic geometry, ordered algebraic structures, partially ordered sets, universal algebra, logic, or theoretical computer science. We hope that we have succeeded in writing a book that can serve these different interests.

Writing a book requires decisions about what to include and what not to include. So far we have described what is included. Now a word about what is not included. There are various classes of topological spaces that are not spectral, but are related to them and serve purposes that are similar to those of spectral spaces (e.g., representation spaces in algebra or the Ziegler spectrum of module categories). We focus our attention (almost) entirely on spectral spaces. But we point out that our topological approach to spectral spaces may sometimes provide substantial clues on how to deal with questions about related, or broader, classes of spaces. Spectral spaces have a richer theory and offer a more coherent panorama than related larger classes of spaces, which we address sporadically at a few points in the book. The applications of spectral spaces are so extensive that they clearly merit an in-depth investigation.

We try to keep the presentation as elementary and concrete as possible. We start with basic definitions and give complete proofs of all results. Concreteness means, first of all, that we include a large number of examples illustrating notions and results, as well as showing how spectral spaces can be produced and handled. Concreteness also means that we are generally not satisfied with just showing that a particular construction with spectral spaces exists. Rather, we always strive to describe the points and the topology of a new space that is produced by a construction. For example, it is not difficult to show that spectral spaces have arbitrary colimits. This statement is a pure existence result. We want to reach a deeper level of understanding and give a detailed description of a colimit (at least in special cases) in terms of the points and topologies of the spaces that are used to produce it.

Despite all efforts to keep the book elementary, concrete, and accessible, the text requires a certain amount of mathematical maturity, as well as basic knowledge in several areas. We name some general references where everything we need and use can be found.

- General topology, [Eng89], [Kel75].
- Algebraic structures such as rings and fields, [AtMa69], [Jac85], [Jac89], [Mat80], [Lan02].
- Partially ordered sets, lattices, and Boolean algebras, [Bly05], [Hal63], [Har05], [Kop89].
- The language of category theory, [HeSt79], [Mac71].
- Notions of domain theory, [GHK+03].
- Model theory, [Hod93].

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Max Dickmann
Niels Schwartz
Marcus Tressl

An Outline of the History of Spectral Spaces

A review of some principal landmarks in the development of the theory of spectral spaces sheds some light on this special part of general topology. It explains the growing general interest in the subject and, in parts, our motivation for writing this book.

Spectral spaces first appeared through the work of Marshall H. Stone in [Sto37b]; one identifiable motivation was the quest for a topological representation of Brouwerian logic. Related to ideas of MacNeille's dissertation, cf. [Mac36] and Birkhoff's [Bir33], Stone extended his celebrated work on the duality between Boolean algebras and Boolean spaces ([Sto36], [Sto37a]) to the realm of distributive lattices.[1] He introduced a topology on the set of prime ideals of a distributive lattice; the resulting space is nowadays called the *spectrum* of the lattice. Stone then singled out certain properties of the spectrum and proved in [Sto37b, Theorems 15,16] that any topological space with these properties is the spectrum of a distributive lattice; this is the birth of spectral spaces and the essence of *Stone duality* between distributive lattices and spectral spaces, cf. Chapter 3.

A spectral space is not Hausdorff, unless it is Boolean. As in Stone's earlier duality between Boolean algebras and Boolean spaces, the distributive lattice associated with a spectral space can be conceived of as an algebra[2] of continuous functions. Explicitly, the Boolean algebra $\mathbf{2} = \{0, 1\}$ is topologized such that the singleton $\{0\}$ is the only nontrivial open set. Then a distributive lattice is the lattice of continuous functions from its spectrum to **2**.

1 To simplify the terminology we assume in this outline of the history that distributive lattices are bounded (i.e., have a smallest and a largest element).

2 By an *algebraic structure*, or an *algebra*, we mean a universal algebra enriched by a set of relations, which is also known (in model theory) as a *first-order structure*.

It was quickly realized that arbitrary rings (not necessarily commutative) can be represented in a related way. Jacobson introduced a topology on the set of primitive ideals of a not necessarily commutative ring, [Jac45, p. 234], and called this the *structure space*. Jacobson refers to [Sto37a] and Gelfand–Silov, cf. [GeŠi41], for the topology he used. Even if he does not use Stone's work in a technical way, he seems to have taken inspiration from Stone. His way of defining the topology follows Stone's lead. Before Jacobson, roughly simultaneously with Stone, the study of rings of continuous functions was under way. This research direction picked up speed in the 1940s, see the bibliography of [GiJe60] (which includes references to some of Stone's work). The idea of representing algebraic structures via something like continuous functions was an active research topic. Possibly starting with Jacobson, this idea took hold in abstract algebra, where rings are represented via structure spaces.

In the case of a commutative unital ring, the primitive ideals are exactly the maximal ideals. Going one step further, one can use the set of all prime ideals to represent a commutative ring. In general, this leads to a more faithful representation. The topologies introduced by Stone and by Jacobson are built on the same principle. These type of topologies were later named *hull–kernel* topologies. By the same method, the *Zariski spectrum* of a commutative unital ring is the set of prime ideals equipped with a natural hull–kernel topology. It is difficult to pinpoint who was first to define this topology (i.e., who actually invented the Zariski spectrum and who first named it after Zariski).[3] For early sources about the spectrum we refer to [Gro60, p. 80] and [Bou61, p. 124], where the word "spectrum" is used and the topology with its main properties is described, in particular showing that the axioms of a spectral space are satisfied.

Another important development, dating from the end of the 1940s, was the introduction (by Zariski, [Zar52, p. 79 f]) of a topology on any classical algebraic variety over a field. (A variety is the common zero set of a set of polynomials.) For each variety, the subvarieties are a basis of closed subsets for a topology, which was later called the *Zariski topology*. In general, this is not a Hausdorff topology. A variety over the real field or the complex field also carries the *Euclidean topology*. The Zariski topology is coarser than the Euclidean topology, but is defined in much greater generality (e.g., in the case of varieties over fields with positive characteristic).

With each classical variety one associates its *coordinate ring* (i.e., its ring

[3] Also it seems to be unknown who first called this space a "spectrum." Stone, who was a functional analyst, saw connections with spectra of operators, as evidenced by his papers [Sto40], [Sto41]. For a simple formal connection between spectra of rings and spectra of operators, consider a finite-dimensional k-vector space V with an operator $T : V \to V$. One can view V as a module over the k-algebra $k[T]$. Then the spectrum of the operator is essentially the same as the Zariski spectrum of $k[T]$.

of polynomial functions). This is akin to the ring of continuous functions on topological spaces. The coordinate ring has a Zariski spectrum. Sending a point of the variety to the maximal ideal of all polynomial functions vanishing at the point, defines a map from the variety to the Zariski spectrum of the coordinate ring. It is injective, and its image is dense in the Zariski spectrum in a very strong sense. The topology of the Zariski spectrum restricts to the Zariski topology on the variety. Thus, the Zariski spectrum of the coordinate ring is essentially the same thing as the variety with its Zariski topology. More explanations can be found in the introduction to Chapter 12 and in 12.3.15.

Considering algebraic geometry from this point of view, a variety consists of a ring of polynomial functions together with its Zariski spectrum. Going one step further in this direction, one can consider arbitrary commutative unital rings with their Zariski spectrum as a generalization of varieties. This leads us to the starting point of Grothendieck's new algebraic geometry. Its basic building blocks are *affine schemes*, namely: each commutative unital ring A, defines a corresponding affine scheme. The underlying space is the Zariski spectrum of the ring, and the rings of sections are locally defined in terms of rings of fractions of A. The given ring A is the ring of global sections of the sheaf. Thus, informally speaking, in this way every commutative ring can be viewed as a ring of functions on its Zariski spectrum. This brings us back to Jacobson and the original idea of representing a ring as a ring of functions.

Apparently Grothendieck never referred to Stone's work on spectral spaces (cf. [Joh86, p. xix]). But, even if the details are not well documented, one can perceive an evolution of ideas from Stone to Jacobson and, eventually, to Grothendieck.

The next landmark came, at the end of the 1960s, with Hochster's dissertation, [Hoc67], [Hoc69]. The functor from rings to spectral spaces sending a ring A to its Zariski spectrum is not a duality, as in the case of distributive lattices. However, Hochster showed that every spectral space is homeomorphic to the Zariski spectrum of some ring, although in a provably non-canonical way. Hochster also coined the name "spectral spaces" for the topological spaces satisfying Stone's axioms, cf. [Hoc69, Introduction].

Hochster's results received a great deal of attention. In view of Grothendieck's work in algebraic geometry it was clear that Zariski spectra, hence spectral spaces, are an important class of topological spaces. There was a growing realization that one can associate spectral spaces with many other kinds of mathematical structures. Often this does not require much more than an adaptation of the construction of spectra of distributive lattices or of Zariski spectra, *mutatis mutandis*. Other constructions include (the list is far from exhaustive):

- Spectra of Abelian ℓ-groups (= lattice-ordered groups) and of f-rings, [Kei71], [BKW77, Chapitre 10] and very much related: spectra of MV-algebras, [BDNS94], [DuPo10].
- The real spectrum of a ring, [CoRo82], [BCR87, Chapitre 7], also treated in Chapter 13 of the present book.
- Differential spectra, for example [Kei77].
- Spectra in logic, for example [Èsa74], [Èsa04], [McTa44], [McTa46], [Lou83].
- Spectra used in sheaf representations of structures, [Die84], [Joh86, Chapter V].
- Valuation spectra of all sorts, for example [HuKn94], [Rob86].

Shortly after Hochster's work, Priestley in [Pri70], gave an alternative description of the representation spaces of distributive lattices. Referring to Stone's characterization, she intended to give "a much simpler characterization in terms of ordered topological space," *loc. cit.*, Introduction. Her representation spaces are Boolean spaces together with a partial order (which we call a *spectral order*) satisfying a compatibility condition. These partially ordered Boolean spaces are now called *Priestley spaces*. Priestley spaces are the same as the representation spaces of Stone (i.e., spectral spaces), but under a somewhat different clothing, cf. [Pri74] and [Fle00]. The achievements of Hochster and Priestley were picked up by mathematicians right away. However, the development progressed in almost disjoint communities. This has not changed much over the years and is a further motivation for writing this book.

Priestley spaces and their connection with spectral spaces are presented in Section 1.5. We use Priestley's version of spectral spaces primarily for the construction of examples.

Starting in the late 1970s, spectral spaces played a key role in real algebraic geometry (i.e., in the study of algebraic varieties defined over the field of real numbers). Real algebraic geometry had existed for a long time, but very much in the shadow of complex algebraic geometry. In the 1970s this began to change slowly. The development gathered momentum and became very dynamic with the introduction of the *real spectrum* of a ring by Coste and Roy (see [CoRo82], as well as references therein, and Chapter 13 of the present book). Similar to the map from a classical variety to the Zariski spectrum of its coordinate ring (see above), every real variety can be mapped injectively and densely into the real spectrum of its coordinate ring. It is a major difference compared with arbitrary varieties and Zariski spectra that the topology of the real spectrum restricts to the Euclidean topology on the variety. Nowadays the real spectrum is an indispensable tool in real algebra and real algebraic geometry.

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Over the past two decades we have seen about 400 publications directly studying, or significantly using, spectral spaces. This was triggered by applications in an enormous range of mathematical subject areas like theoretical computer science, category theory, functional analysis, representation theory, to name just a few. In addition to these there are the classical core areas of topology, lattices, ring theory, algebraic geometry, and logic.

We hope that the present book will give a common ground to the subject and help readers to navigate its wider ramifications.