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Spectral Spaces and Spectral Maps

Spectral spaces are a class of topological spaces. They were first described in terms of three topological conditions (but not yet named) in [Sto37b, Theorems 15, 16]. A slightly different, but equivalent, axiomatization was given in [Hoc67], [Hoc69], where the name *spectral space* was also introduced. Hochster used four axioms, which we present and discuss in Section 1.1. The axioms contain conditions familiar to every mathematician – the separation axiom T_0 and quasi-compactness. The other conditions say that there is a distinguished basis of open sets and that the nonempty closed and irreducible subsets correspond to the points of the space.

Spectral spaces can be related to each other via continuous maps, since they are topological spaces. But arbitrary continuous maps between spectral spaces do not connect the distinguished bases of domain and codomain with each other, thus disregarding decisive features of the spaces. Therefore, in Section 1.2, a more suitable class of maps is introduced, the *spectral maps*. The spectral spaces and the spectral maps together form a category, which provides an excellent framework for the further study of spectral spaces.

Spectral spaces carry a great deal of structure – besides the defining topology there are two other topologies, the *patch topology*, also called the *constructible topology*, cf. Section 1.3, and the *inverse topology*, cf. Section 1.4.

In every topological space a binary relation, called *specialization*, is defined by: $x \rightsquigarrow y$ if and only if the neighborhood filter of y is contained in the neighborhood filter of x. In the case of T_0-spaces, hence of spectral spaces, the relation is a partial order. In Section 1.5 the specialization order is analyzed in the context of spectral spaces. A spectral space is uniquely determined by its patch topology and the specialization order. This fact is the basis for another approach to spectral spaces, which is due to H. Priestley and leads to the notion of a *Priestley space*, see [Pri70] and numerous other publications. In Section 1.5 we show that spectral spaces and Priestley spaces are the same mathematical structures, just viewed differently. The focus of the book is on the topology of spectral spaces, but Priestley spaces are a particularly valuable tool for the construction of examples.

Various simple spectral spaces can be produced *ad hoc* and are presented throughout to illustrate the basic notions. The chapter closes with Section 1.6, where a detailed presentation of several examples and constructions is given, highlighting the different structural features of spectral spaces developed in the preceding sections.

1.1 The Definition of Spectral Spaces

Summary Spectral spaces are a class of topological spaces defined by four axioms, see 1.1.5. We analyze each of the axioms on its own and explore the consequences of combinations of different axioms. The definition, as well as the characterization given in 1.1.14, indicate that the quasi-compact open sets play a key role in spectral spaces.

In 1.1.15 it is shown that every finite T_0-space is spectral, which gives us a first collection of examples.

To start with, it is necessary to explain some terminology and notation that is used throughout. References for basic facts from general topology are [Bou71b], [Eng89] or [Kel75].

1.1.1 Some Notation and Terminology Let X be a topological space. The set of open subsets (i.e., the topology) is denoted by $\mathcal{O}(X)$; the set of closed subsets is denoted by $\mathcal{A}(X)$. Both $\mathcal{O}(X)$ and $\mathcal{A}(X)$ contain $\emptyset$, X, and are closed under finite unions and finite intersections. Thus, $\mathcal{O}(X)$ and $\mathcal{A}(X)$ are bounded sublattices[1] of the Boolean algebra $\mathfrak{P}(X)$, the power set of X. It follows that they are distributive lattices. Moreover, $\mathcal{O}(X)$ is closed under arbitrary unions and $\mathcal{A}(X)$ is closed under arbitrary intersections. Thus, they are even complete lattices – but usually infinite meets in $\mathcal{O}(X)$ and infinite joins in $\mathcal{A}(X)$ do not coincide with those in $\mathfrak{P}(X)$.

Our topological spaces are *typically not Hausdorff*. Therefore some care is needed with regard to the terminology we use. In the literature a large part of the terminology for topological spaces is adapted to the needs of analysis and assumes the Hausdorff separation axiom. Whenever we use names that may cause confusion, we shall always explain the way we use them.

[1] That is, sublattices containing the smallest and largest elements of $\mathfrak{P}(X)$.

1.1.2 Quasi-Compact Sets A topological space is **quasi-compact** if every open cover has a finite subcover. A space is **compact** if it is quasi-compact and Hausdorff.[2]

Now let X be any topological space. A subset S of X is quasi-compact if it is quasi-compact in its relative topology. Finite unions of quasi-compact subsets of X are quasi-compact, but finite intersections need not be quasi-compact. The subsets of X that are at the same time open and quasi-compact play a crucial role in this book. We call them **quasi-compact open** sets and write

$$\mathring{\mathcal{K}}(X) := \{O \subseteq X \mid O \text{ is quasi-compact open}\}.$$

If τ denotes the topology of X, we also write $\mathring{\mathcal{K}}(\tau)$ instead of $\mathring{\mathcal{K}}(X)$. Note that $\emptyset \in \mathring{\mathcal{K}}(X)$ for any topological space X. There are many spaces, in particular in classical analysis, having no other quasi-compact open sets. The set $\mathring{\mathcal{K}}(X)$ is closed under finite unions, hence is a join-subsemilattice of $\mathfrak{P}(X)$.

1.1.3 Specialization Every topological space X carries a quasi-order,[3] which is called the **specialization order** and is defined by:

$$x \rightsquigarrow y \text{ if and only if } y \in \overline{\{x\}}.^4$$

We say that y is a **specialization** of x, and x is a **generalization** of y. If the topology is denoted by τ then we also write $\underset{\tau}{\rightsquigarrow}$.

Specialization can be checked using any subbasis $\mathcal{S}$ of open sets: $x \rightsquigarrow y$ if and only if $y \in O$ and $O \in \mathcal{S}$ imply $x \in O$.

A subset $A \subseteq X$ is **closed under specialization**, or **specialization-closed**, if $x \in A$ and $x \rightsquigarrow y$ implies $y \in A$. The set is **closed under generalization**, or **generically closed**, if $y \in A$ and $x \rightsquigarrow y$ implies $x \in A$. Closed subsets are closed under specialization, open subsets are closed under generalization.

Any set $A \subseteq X$ is contained in a smallest set closed under specialization,

$$\text{Spez}(A) = \{y \in X \mid \exists x \in A\colon x \rightsquigarrow y\} = \bigcup_{x \in A} \overline{\{x\}},$$

and in a smallest set closed under generalization,

$$\text{Gen}(A) = \{x \in X \mid \exists y \in A\colon x \rightsquigarrow y\}.$$

2 The reader should be aware that there is no agreement in the literature on whether “compact spaces” are Hausdorff or not.
3 A quasi-order is a reflexive and transitive binary relation and need not be antisymmetric, A.1(i).
4 There is no uniform way in the literature to define specialization, cf. 7.1.7. See 12.1.14 for our motivation.

If $A = \{a\}$ is a singleton set then we also write $\mathrm{Spez}(A) = \mathrm{Spez}(a)$ and $\mathrm{Gen}(A) = \mathrm{Gen}(a)$.

Continuous maps preserve specialization (i.e., if $f: X \to Y$ is continuous and $x \rightsquigarrow x'$ in X then $f(x) \rightsquigarrow f(x')$ in Y).

Suppose that X carries two topologies σ and τ. If $\sigma \subseteq \tau$ then the specialization relation $\underset{\sigma}{\rightsquigarrow}$ is stronger than $\underset{\tau}{\rightsquigarrow}$ (i.e., $x \underset{\tau}{\rightsquigarrow} x'$ implies $x \underset{\sigma}{\rightsquigarrow} x'$).

1.1.4 Reminder (a) Suppose X is a set and $\mathcal{S} \subseteq \mathfrak{P}(X)$. The set $\mathcal{S}$ **separates points** in X if for all $x \neq y$ in X there is some $S \in \mathcal{S}$ that contains exactly one of the points (we do not specify which one!). In particular, a topological space X is a T_0**-space** if $\mathcal{O}(X)$ separates points in X.[5]
(b) A subset C of a topological space X is **irreducible** if for all closed subsets $A, B \subseteq X$ with $C \subseteq A \cup B$ we have $C \subseteq A$ or $C \subseteq B$. Notice that a set is irreducible if and only if its closure is irreducible. Thus, C is irreducible if and only if $\overline{C}$, as an element of the bounded distributive lattice $\mathcal{A}(X)$, is join-irreducible (cf. A.6(vii)). Clearly, (closures of) singletons are irreducible.

1.1.5 Definition A **spectral space** is a topological space X that satisfies the following four conditions.

- **S1**: X is quasi-compact and T_0.
- **S2**: $\mathring{\mathcal{K}}(X)$ is a basis of open subsets of X.
- **S3**: The intersection of two quasi-compact open subsets of X is again quasi-compact.
- **S4**: X is **sober**, that is, for every nonempty closed and irreducible subset C of X, there is a point $x \in X$, necessarily unique (by **S1**), with $C = \overline{\{x\}}$.

The topology of X is called the **spectral topology**.

We start by looking at the conditions in Definition 1.1.5 separately, discuss some basic facts, and record their impact on the set $\mathring{\mathcal{K}}(X)$.

1.1.6 The T_0 Property It is easily checked that a topological space X has the T_0-property if and only if specialization is a *partial order* (i.e., is an antisymmetric quasi-order). Thus, a spectral space is partially ordered by specialization.

The specialization relation is irrelevant for many spaces in classical analysis, for these are mostly T_1-spaces,[6] and one characterization of T_1-spaces says that every singleton is a closed set. Thus, the specialization order of a T_1-space is the trivial partial order – every point is comparable only with itself. Usually

[5] T_0-spaces are also called **Kolmogorov spaces**.
[6] Recall that a topological space is a T_1**-space** if, given two distinct points, each one has a neighborhood not containing the other one.

spectral spaces are not T_1-spaces, and we shall see that the specialization order is one of their essential features.

Thus, partially ordered sets (also called posets, A.1(iii)) play an essential role throughout this book. For notation and terminology, we refer the reader to the Poset Zoo in the Appendix.

1.1.7 On Bases of Topological Spaces Suppose X is an arbitrary topological space and $\mathcal{L} \subseteq \mathcal{O}(X)$ is a basis of the topology closed under finite unions (e.g., $\mathcal{L}$ could be $\overset{\circ}{\mathcal{K}}(X)$ if **S2** is satisfied, cf. 1.1.2, 1.1.5). Thus, $\mathcal{L}$ is a join-subsemilattice of $\mathcal{O}(X)$. If $O \subseteq X$ is open, then the set

$$i(O) := \{U \in \mathcal{L} \mid U \subseteq O\}$$

is an ideal of the join-semilattice $\mathcal{L}$ (i.e., $i(O)$ is closed under finite unions and, if $U \subseteq U' \in i(O)$ and $U, U' \in \mathcal{L}$, then $U \in i(O)$, cf. A.7(i)). The hypothesis that $\mathcal{L}$ is a basis allows us to recover O from $i(O)$. For then, every open set O can be written as

$$(*) \qquad O = \bigcup_{U \in i(O)} U.$$

Let $\mathcal{I}(\mathcal{L})$ be the set of ideals of $\mathcal{L}$. Then the definition of $i(O)$ yields a map

$$i\colon \mathcal{O}(X) \to \mathcal{I}(\mathcal{L}).$$

Now $(*)$ says that the function

$$j\colon \mathcal{I}(\mathcal{L}) \to \mathcal{O}(X);\quad I \mapsto \bigcup I = \bigcup_{U \in I} U,$$

satisfies $j \circ i = \mathrm{id}_{\mathcal{O}(X)}$. In fact, $i(O)$ is the largest subset $\mathcal{S} \subseteq \mathcal{L}$ with $O = \bigcup \mathcal{S}$. Obviously, $i \circ j(I) \supseteq I$ for all $I \in \mathcal{I}(\mathcal{L})$. In general, this inclusion is proper. For example, let $X \subseteq \mathbb{R}$ be the closed unit interval, take $\mathcal{L} = \mathcal{O}(X)$, and let $I \subseteq \mathcal{L}$ be the ideal of open subsets of X whose closure does not contain 0. Then $j(I) = (0, 1] \in i \circ j(I) \setminus I$.

1.1.8 On Axiom S2 Let X be a topological space and suppose that $\mathcal{B}$ is a basis of the topology, contained in $\overset{\circ}{\mathcal{K}}(X)$ and closed under finite unions (including the empty union, which is $\emptyset$). Then every $U \in \mathcal{O}(X)$ is a union of sets from $\mathcal{B}$. By quasi-compactness, every $V \in \overset{\circ}{\mathcal{K}}(X)$ is a *finite* union of sets from $\mathcal{B}$, hence is a member of $\mathcal{B}$. We conclude that $\mathcal{B} = \overset{\circ}{\mathcal{K}}(X)$.

Thus, if X has a basis contained in $\overset{\circ}{\mathcal{K}}(X)$, then there is a unique one that is closed under finite unions, namely $\overset{\circ}{\mathcal{K}}(X)$.

1.1.9 Proposition *Suppose that X is a topological space satisfying* **S2** *(i.e., $\mathring{\mathcal{K}}(X)$ is a basis of the topology). Then $j\colon \mathcal{I}(\mathring{\mathcal{K}}(X)) \to \mathcal{O}(X)$ and $i\colon \mathcal{O}(X) \to \mathcal{I}(\mathring{\mathcal{K}}(X))$ are mutually inverse bijective maps.*

Proof As $\mathring{\mathcal{K}}(X)$ is closed under finite unions, we can apply 1.1.7. For each $I \in \mathcal{I}(\mathring{\mathcal{K}}(X))$ the inclusion $I \subseteq i \circ j(I)$ holds trivially. It remains to show that $i \circ j(I) \subseteq I$. So, pick an element $U \in i \circ j(I)$ (i.e., $U \in \mathring{\mathcal{K}}(X)$ and $U \subseteq j(I) = \bigcup I$). As U is quasi-compact there is a finite subset $J \subseteq I$ such that $U \subseteq \bigcup J$. The ideal I is closed under finite unions, hence $\bigcup J \in I$. Thus U is contained in an element of I, and, since I is an ideal, we get $U \in I$, as claimed. □

1.1.10 On Axiom S3 Observe that we have not invoked axiom **S3** so far. Now suppose that a topological space X satisfies **S3**. Thus, $\mathring{\mathcal{K}}(X)$ is a sublattice of $\mathcal{O}(X)$ since it is always closed under finite unions, 1.1.2, and, assuming **S3**, is also closed under finite intersections. If, in addition, X is quasi-compact (e.g., if X is a spectral space), then $\mathring{\mathcal{K}}(X) \subseteq \mathcal{O}(X)$ is even a bounded sublattice.

Given a spectral space X, the lattice $\mathring{\mathcal{K}}(X)$ is an important part of its structure. It builds a bridge between spectral spaces and lattices. In Chapter 3 we shall see that this connection is much closer than the present considerations show. In fact, the spectral space X can be fully recovered from the lattice $\mathring{\mathcal{K}}(X)$ alone.

Assuming the axioms **S1**, **S2**, and **S3**, the next result shows that the bijective correspondence of 1.1.9 between the ideals of $\mathring{\mathcal{K}}(X)$ and the open sets of X restricts to a bijection between the complements of nonempty closed and irreducible subsets of X and the prime ideals of the lattice $\mathring{\mathcal{K}}(X)$.

1.1.11 Proposition *Suppose that X is a topological space and $\mathcal{L} \subseteq \mathcal{O}(X)$ is a bounded sublattice and a basis of the topology. Let $C \subseteq X$ be a closed set and set*

$$I = i(X \setminus C) = \{U \in \mathcal{L} \mid U \cap C = \emptyset\}.$$

Then I is an ideal of $\mathcal{L}$ and

(i) *$C \neq \emptyset$ if and only if I is a proper ideal (i.e., $I \neq \mathcal{L}$).*
(ii) *C is a nonempty and irreducible set if and only if I is a prime ideal (i.e., $I \neq \mathcal{L}$ and for all $U, V \in \mathcal{L}$ with $U \cap V \in I$ we have $U \in I$ or $V \in I$).*

Proof By 1.1.7 we know that I is an ideal of $\mathcal{L}$.

(i) Clearly, $I = \mathcal{L}$ if $C = \emptyset$. Conversely, $I = \mathcal{L}$ implies $X \in I$, hence $C = X \cap C = \emptyset$.

(ii) First suppose that C is irreducible and take $U, V \in \mathcal{L}$ with $U \cap V \in I$. Then $U \cap V \cap C = \emptyset$ and so $C \subseteq (X \setminus U) \cup (X \setminus V)$. As C is irreducible we have $C \subseteq X \setminus U$ or $C \subseteq X \setminus V$ (i.e., $U \in I$ or $V \in I$).

Conversely, suppose C is not irreducible. Take closed sets $A_1, A_2 \subseteq X$ with $C \nsubseteq A_i$, but $C \subseteq A_1 \cup A_2$. There are points $c_i \in C \cap (X \setminus A_i)$ and neighborhoods $U_i \in \mathcal{L}$ of c_i with $U_i \subseteq X \setminus A_i$. We see that $U_i \notin I$. On the other hand, the inclusion $U_1 \cap U_2 \subseteq (X \setminus A_1) \cap (X \setminus A_2)$ shows that

$$(U_1 \cap U_2) \cap C \subseteq U_1 \cap U_2 \cap (A_1 \cup A_2) = \emptyset.$$

Thus $U_1 \cap U_2 \in I$, and I is not prime. □

1.1.12 Corollary *Let X be a topological space satisfying axioms* **S1–S3**. *Then the map*

$$\begin{aligned}\{C \in \mathcal{A}(X) \mid C \neq \emptyset \text{ irreducible}\} &\to \{I \in \mathcal{I}(\mathring{\mathcal{K}}(X)) \mid I \text{ prime}\}\\ C &\mapsto i(X \setminus C)\end{aligned}$$

is bijective. □

1.1.13 On Axiom S4 and Soberness Suppose that X is a topological space. Every set $\overline{\{x\}}$ is closed, irreducible, and nonempty. If $A = \overline{\{x\}}$, then x is called a generic point of A, cf. A.2. In general, a nonempty closed and irreducible set need not have a generic point and, if it has one, there might be several of them (e.g., if X is an indiscrete space with at least two points). Axiom **S4** says that the sets $\overline{\{x\}}$ are the only nonempty closed and irreducible sets.

A space is T_0 if and only if every closed and irreducible set A has *at most one* generic point. Thus, in this case A has a smallest element for specialization. Hence the points of a sober T_0-space are in bijection with the nonempty closed and irreducible sets.

1.1.14 Conclusion The preceding considerations yield the following alternative characterization of spectral spaces, which highlights the key role played by the lattice $\mathring{\mathcal{K}}(X)$. A topological space X is spectral if and only if

(i) the set $\mathring{\mathcal{K}}(X)$ is a bounded sublattice of $\mathcal{O}(X)$, separates points of X, is a basis of the topology, and

(ii) for every prime ideal $I \subseteq \mathring{\mathcal{K}}(X)$ there is a unique point $x \in X$ such that $I = \{U \in \mathring{\mathcal{K}}(X) \mid x \notin U\}$.

In 1.1.10 we announced that a spectral space X is completely determined by the lattice $\mathring{\mathcal{K}}(X)$. This will be proved in Chapter 3. The present characterization is a first step in this direction. It shows how the points of X can be reconstructed from the lattice $\overset{\circ}{\mathcal{K}}(X)$.

We exhibit a collection of first examples of spectral spaces. They are all finite spaces. Trivially, if X is a finite topological space then every subset is quasi-compact, hence $\overset{\circ}{\mathcal{K}}(X) = \mathcal{O}(X)$. Only the T_0-property and soberness have to be discussed.

1.1.15 Proposition *Every finite T_0-space is spectral.*

Proof It suffices to show soberness (axiom **S4**). So, let C be a nonempty closed and irreducible set. Since C is finite, the set

$$\bigcup_{c\in C} \overline{\{c\}}$$

is closed, hence equals C. As C is irreducible it cannot be covered by finitely many proper closed subsets. Therefore there is some $c \in C$ with $C = \overline{\{c\}}$, as required. □

1.1.16 Finite Spectral Spaces vs. Finite Posets According to 1.1.15 the finite T_0-spaces are exactly the finite spectral spaces. Moreover, we emphasize that the finite T_0-spaces essentially coincide with the finite posets. One assigns the underlying specialization poset to a finite T_0-space. Conversely, one notes that a finite poset $(P, \leq)$ carries a unique T_0-topology having specialization order $\leq$. Namely, the open sets are exactly the down-sets (i.e., the fine lower topology of $(P, \leq)$, which coincides with the coarse lower topology, has this property; see Poset Zoo, A.8).

1.1.17 Example There is a unique topology on the empty set. The topology is trivially T_0, hence $\emptyset$ is a spectral space.

Every singleton carries a unique topology, which, again, is trivially T_0, hence makes the set a spectral space. We write $\mathbb{1}$ for the spectral space with underlying set $\mathbf{1} = \{0\}$.

1.1.18 Example The set $\mathbf{2} = \{0, 1\}$ carries four different topologies. The only one that is not T_0 is the indiscrete topology. The other three topologies are spectral. Explicitly:

(i) The discrete topology is spectral. The specialization order is trivial (i.e., every element is comparable only with itself).

(ii) The fine lower topology for the total order $0 < 1$ has the open sets $\emptyset, \{0\}, \mathbf{2}$. The set $\mathbf{2}$ with this topology is called the **Sierpiński space** and is denoted by $\mathbb{2}$. The Sierpiński space is spectral, the point 0 is isolated and not closed, whereas the point 1 is closed and not isolated. The specialization order is the natural total order, that is, $0 \rightsquigarrow 1$.

(iii) Interchanging the role of 0 and 1 in the Sierpiński space (i.e., starting with the total order $1 < 0$), we obtain a spectral space with universe **2** and specialization relation $1 \rightsquigarrow 0$.

1.1.19 Example Given $n \in \mathbb{N}$, the set $\mathbf{n} = \{0, \dots, n-1\}$ carries many topologies (except for the cases discussed in the previous examples), but we are interested in only three of them.

(i) The discrete topology is spectral, again with trivial specialization order.

(ii) The natural total order yields a T_0-topology whose open sets are the intervals $\{0, \dots, k-1\}$, $0 \le k \le n$. The set **n** with this topology is denoted by $\mathbb{n}$. The space $\mathbb{n}$ is spectral and the specialization order is the natural total order, that is

$$0 \rightsquigarrow 1 \rightsquigarrow \cdots \rightsquigarrow n-1.$$

(iii) Reversing the natural order in **n** we obtain a spectral space with universe **n** and specialization relation

$$n-1 \rightsquigarrow n-2 \rightsquigarrow \cdots \rightsquigarrow 0.$$

In fact, any topology on **n** with total specialization order is homeomorphic to $\mathbb{n}$ via a suitable permutation of $\{0, \dots, n-1\}$.

1.1.20 Saturated and Coherent Sets We shall encounter topological spaces that have some properties in common with spectral spaces, but are not necessarily (or not *a priori*) spectral. One important such property is coherence. The definition of coherence requires the following notion.

Let X be a T_0-space with topology τ. A subset $Q \subseteq X$ is called **saturated** if Q is an intersection of open sets. As X is a T_0-space this is the same as saying that Q is generically closed (i.e., $x \rightsquigarrow q \in Q$ implies $x \in Q$, 1.1.3).[7] Note that open sets are generically closed, hence so are their intersections. Conversely, if Q is generically closed then $Q = \bigcap_{x \notin Q} X \setminus \overline{\{x\}}$, hence Q is saturated.

A saturated set Q is called **quasi-compact saturated** if it is also quasi-compact. In every topological space there are plenty of quasi-compact saturated sets. Namely, for each $x \in X$, the set $\mathrm{Gen}(x)$ of generalizations of x is quasi-compact saturated.

[7] Equivalently, Q is a down-set of the poset $(X, \underset{\tau}{\rightsquigarrow})$, or is open for the fine lower topology, A.8(ii).

Claim Assume that $\mathring{\mathcal{K}}(X)$ is a subbasis of open sets for X. Then every quasi-compact saturated set is an intersection of quasi-compact open sets.

Proof of Claim Let $Q \subseteq X$ be quasi-compact saturated and pick $x \in X \setminus Q$. Then $X \setminus \overline{\{x\}} = \bigcup_{i\in I} \bigcap_{k\in F_i} U_{ik}$, where the F_i are finite and the U_{ik} belong to $\mathring{\mathcal{K}}(X)$. For each i there is some $k_i \in F_i$ with $x \notin U_{ik_i}$. Thus, $Q \subseteq \bigcup_{i\in I} U_{ik_i} \subseteq X \setminus \overline{\{x\}}$, and, by quasi-compactness of Q, there is a finite subset $J \subseteq I$ with $Q \subseteq \bigcup_{i\in J} U_{ik_i} \subseteq X \setminus \overline{\{x\}}$. Finite unions of quasi-compact open sets are quasi-compact open. Thus, $\bigcup_{i\in J} U_{ik_i} \in \mathring{\mathcal{K}}(X)$, proving the claim. ⋄

Later we shall see that in a spectral space the quasi-compact saturated sets are precisely the subsets of X that are closed for the *inverse topology*, cf. 1.4.7, also see 1.5.5 and 4.1.6.

The saturated sets are closed under arbitrary unions and intersections, whereas, in general, quasi-compact sets are only closed under finite unions and not under intersections. Thus, finite unions of quasi-compact saturated sets are quasi-compact saturated, but intersections need not be quasi-compact saturated. Therefore the following definition is introduced: the space X is **coherent** if the intersection of two quasi-compact saturated sets is again quasi-compact saturated.

Spectral spaces are coherent. This is immediate from the fact (mentioned above) that the quasi-compact saturated sets are exactly the subsets closed for the inverse topology. So, we could have replaced the spectral space axiom **S3** by the condition that coherence holds.

1.2 Spectral Maps and the Category of Spectral Spaces

Summary Spectral maps are the appropriate tool to describe connections between different spectral spaces. Spectral maps are continuous, but satisfy a stronger requirement, see 1.2.2 and, for motivation, 1.2.1. The spectral spaces together with the spectral maps form the *category of spectral spaces*, cf. 1.2.3, which is denoted by `Spec`. The category plays a key role in the development of the theory and in many applications, see Section 2.5 for first examples. Properties of `Spec` are an important topic throughout the book.

The notion of a spectral map is illustrated by examples involving finite spectral spaces. The examples are used to obtain first pieces of information about the category `Spec`, 1.2.7 and 1.2.8.