A Student’s Guide to Analytical Mechanics

Analytical mechanics is a set of mathematical tools used to describe a wide range of physical systems both in classical mechanics and beyond. It offers a powerful and elegant alternative to Newtonian mechanics; however, it can be challenging to learn due to its high degree of mathematical complexity. Designed to offer a more intuitive guide to this abstract topic, this guide explains the mathematical theory underlying analytical mechanics; helping students formulate, solve, and interpret complex problems using these analytical tools. Each chapter begins with an example of a physical system to illustrate the theoretical steps to be developed in that chapter, and ends with a set of exercises to further develop students’ understanding. The book presents the fundamentals of the subject in depth before extending the theory to more elaborate systems, and includes a further reading section to ensure that this is an accessible companion to all standard textbooks.

JOHN L. BOHN is a professor of physics at the University of Colorado Boulder. He is a fellow of JILA – an interdisciplinary institute for quantum physics, chemistry, and astronomy – and a fellow of the American Physical Society.

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Cambridge University Press
978-1-107-14576-4 — A Student's Guide to Analytical Mechanics
John L. Bohn
Frontmatter
More Information

A Student’s Guide to Analytical Mechanics

John L. Bohn
University of Colorado Boulder

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107145764
DOI: 10.1017/9781316536124

First published 2018

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Bohn, John L., 1965– author.
Title: A student's guide to analytical mechanics /
John L. Bohn (University of Colorado Boulder).
Other titles: Analytical mechanics
Description: Cambridge : Cambridge University Press, 2018. |
Includes bibliographical references and index.
Identifiers: LCCN 2018010114 | ISBN 9781107145764 (hardback) |
ISBN 9781316509074 (pbk.)
Subjects: LCSH: Mechanics, Analytic–Textbooks.
Classification: LCC QA807 .B64 2018 | DDC 531.01/515–dc23 LC record
available at https://lccn.loc.gov/2018010114

ISBN 978-1-107-14576-4 Hardback
ISBN 978-1-316-50907-4 Paperback

Additional resources for this publication at www.cambridge.org/9781107145764.

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Preface

There is no intellectual exercise that is not ultimately pointless.
—J. L. Borges, in “Pierre Menard, Author of the *Quixote*”

Classical mechanics originated in the qualitative speculations of the ancient philosophers. And look where it got them: they were uncertain whether the heavy rock fell faster than the light one. Later this situation was rectified by the efforts of many thinkers studying various simple examples to extract the general rules for the behavior of mechanical things. This effort may be said to have converged on Newton, who articulated the fundamental principle of mechanics in $F = ma$. After Newton, the subject diverged again, introducing various new mathematical ways to look at the equations of motion, yet always based ultimately on the same physical content.

The main mathematical development, and the one that is the topic of this book, is analytical mechanics, by which I mean Lagrangian and Hamiltonian mechanics. These are wonderfully elegant, concise, and – in modern treatments – *abstract* formulations of mechanics. To the student, the level of sophistication of these theories is a two-edged sword. On one edge (probably the sharp one), one can formulate simply and axiomatically some very powerful tools for setting up and solving problems, which then expand the reach of the student’s powers and lead to applications in quantum theory and statistical mechanics. However, on the dull edge of the sword, these axioms are seemingly arbitrary and divorced from the physical reality that gave them life in the first place.

It is on the dull edge of the sword that this book falls. My goal is to illustrate some of the beautiful mathematical ideas of analytical mechanics, using very simple specific mechanical systems. These are things like inclined planes and pendulums, springs and projectiles, whose motions you can easily imagine and understand; you could even build a lot of these things. In this way I intend to get back to the basics of mechanics as a tool for formulating, solving, and interpreting simple problems in mechanics. Later on, the student will recognize these same ideas as applied in other areas of physics.

This approach is not unlike the quest of Borges' hero Menard, in the short story quoted above. Menard set himself the task of reproducing the text of *Don Quixote* despite being a twentieth-century author. His goal was not to copy the famous book, but to put himself in he mindset of its author, so that the text of the novel would come to him naturally. In a similar way, I hope to inhabit the mindset of someone familiar with $F = ma$ but seeking a more elegant way to apply it. This perspective will build upon familiar concepts to see that the new ones are plausible.

Put it another way: there seem to be two camps of students in theoretical physics. One camp possesses complete comfort with mathematical abstraction. Such a reader sees in the mathematical development an intrinsic rightness and beauty, which makes particular examples at least easy, at worst distracting. The second camp is more comfortable considering concrete examples of real-world things, so that when the mathematics is applied, it is clear what this mathematics is supposed to represent. This is the difference between deductive reasoning and inductive reasoning. It is at this latter group of inductive thinkers that the present book is aimed.

This is a task not without its hazards. To present the subject with the crystalline precision of mathematics implies that there is pretty much a right way to describe things. On the other hand, presenting ways of thinking about the math can be awfully subjective, and the ones I propose here might not sit well with all readers. My quest might be as pointless as Menard's. But this is okay; perhaps the important thing is to convince you that this kind of interpretation is both possible and desirable, and to encourage you to seek one that suits your own talents.

I gratefully acknowledge the support of the JILA-NSF Physics Frontier Center and the National Institute of Standards and Technology during the preparation of this work.