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1

Free Fields in Vacuum

This chapter introduces notations and obtains results for free fields in vacuum that we shall utilise in the rest of the book. In applications, one generally considers particles of spin 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$, which are conventionally described respectively by scalar, Dirac, vector and Rarita–Schwinger fields. Accordingly, we discuss each of these fields separately and derive their spectral functions and Feynman propagators. But we shall not limit ourselves to these fields only, and we will extend their formal constructions to general fields describing particles of arbitrary spin.

The conventional approach to quantum field theory begins with the Lagrangian, from which a Klein–Gordon equation is derived for each field component. These are then expanded in terms of one-particle annihilation and creation operators. There is also the other approach, expounded by Weinberg [1], that starts from particle states and constructs fields requiring causality and Lorentz covariance. In this brief review we exploit both points of view, starting with the Lagrangian for low-spin fields and with the one-particle states for high-spin fields. In either case, we do not need to construct explicitly the coefficient functions in the expansion of fields. As we shall be interested only in the spectral functions and propagators, all we require are the spin sums over the product of such functions. These sums may be obtained directly by using the supplementary conditions eliminating components in excess of those needed to describe different spin degrees of freedom.

We begin with general fields to present a unified view of some properties of free fields. Then, in the following sections, we treat separately the individual fields of low spin and show how the supplementary conditions on the field suffice to evaluate the spin sums. We come back to general fields in the last section.

1.1 Generalities

Let $|\boldsymbol{q}, \sigma\rangle$ be the state vector of a particle of mass m, momentum $\boldsymbol{q}$ and spin j with z-component σ in its rest frame ($\sigma = j, j-1, \cdots, -j$). It is normalised in a Lorentz invariant way:

$$\langle \boldsymbol{q}', \sigma' | \boldsymbol{q}, \sigma \rangle = (2\pi)^3 2\omega \delta_{\sigma\sigma'} \delta^3(\boldsymbol{q} - \boldsymbol{q}'), \qquad \omega = +\sqrt{\boldsymbol{q}^2 + m^2}. \tag{1.1.1}$$

Define as usual the creation operator $a^\dagger(\boldsymbol{q}, \sigma)$ to produce this state by its action on the vacuum state

$$|\boldsymbol{q}, \sigma\rangle = a^\dagger(\boldsymbol{q}, \sigma)|0\rangle, \tag{1.1.2}$$

and the annihilation operator $a(\boldsymbol{q}, \sigma)$ by its adjoint with $a(\boldsymbol{q}, \sigma)|0\rangle = 0$. Then (1.1.1) gives the commutation/anticommutation relation

$$[a(\boldsymbol{q}, \sigma), a^\dagger(\boldsymbol{q}', \sigma')]_\mp = (2\pi)^3 2\omega \delta_{\sigma\sigma'} \delta^3(\boldsymbol{q} - \boldsymbol{q}'). \tag{1.1.3}$$

As always, the top and bottom signs refer to bosonic and fermionic degrees of freedom. Thus the subscript $\mp$ indicates commutator or anticommutator according to whether the particles destroyed and created by a and $a^\dagger$ are bosons or fermions respectively. If the field also describes a distinct antiparticle, we denote the corresponding destruction and creation operator by $b(\boldsymbol{q}, \sigma)$ and $b^\dagger(\boldsymbol{q}, \sigma)$ satisfying

$$[b(\boldsymbol{q}, \sigma), b^\dagger(\boldsymbol{q}', \sigma')]_\mp = (2\pi)^3 2\omega \delta_{\sigma\sigma'} \delta^3(\boldsymbol{q} - \boldsymbol{q}'), \tag{1.1.4}$$

all other commutatators/anticommutators being zero.

A multicomponent field $\psi_l(x)$ is needed to describe particles of non-zero spin. In the conventional description, the index l denotes one or more (Lorentz) vector indices for bosonic fields and an additional (Dirac) spinor index for fermionic fields. Such a description introduces extra components, which are then eliminated by imposing supplementary conditions on $\psi_l(x)$. (The index structure and supplementary conditions are reviewed in Appendix A.) We expand the field in terms of annihilation and creation operators [1]:

$$\psi_l(x) = \int \frac{d^3q}{(2\pi)^3 2\omega} \sum_\sigma \left(u_l(\boldsymbol{q}, \sigma) e^{-iq\cdot x} a(\boldsymbol{q}, \sigma) + v_l(\boldsymbol{q}, \sigma) e^{iq\cdot x} b^\dagger(\boldsymbol{q}, \sigma) \right). \tag{1.1.5}$$

Here and below, four-momenta q^μ with space components integrated over, are generally on mass-shell, $q^2 = m^2$, with q^0 given by the positive square root, $q^0 = \sqrt{\boldsymbol{q}^2 + m^2}$. (For real fields describing a self-charge-conjugate particle, $a \equiv b$.) The coefficient functions u_l and v_l are polarisation spin-tensors in the general case. With the normalisation of creation and annihilation operators already fixed by (1.1.3) and (1.1.4), that of the coefficient functions is determined by the normalisation of the field.

In this chapter we are mainly interested in calculating for different fields the commutator/anticommutator

$$\Delta_{ll'}(x, x') \equiv [\psi_l(x), \psi^\dagger_{l'}(x')]_\mp, \tag{1.1.6}$$

and the Feynman propagator

$$\Delta^F_{ll'}(x, x') \equiv i\langle 0|T\{\psi_l(x)\psi^\dagger_{l'}(x')\}|0\rangle, \tag{1.1.7}$$

where T is the time-ordering symbol

$$T\{\psi_l(x)\psi^\dagger_{l'}(x')\} = \theta(t-t')\psi_l(x)\psi^\dagger_{l'}(x') \pm \theta(t'-t)\psi^\dagger_{l'}(x')\psi_l(x). \tag{1.1.8}$$

Here $\theta(t)$ is a step function, being equal to $+1$ for $t > 0$ and zero for $t < 0$. Inserting the expansion (1.1.5) for ψ in Δ and Δ^F, both reduce to commutators/anticommutators of creation and annihilation operators. Then using (1.1.3) and (1.1.4) we get

$$\Delta_{ll'}(x,x') = \int \frac{d^3q}{(2\pi)^3 2\omega}\left(e^{-iq\cdot(x-x')}\sum_\sigma u_l(\boldsymbol{q},\sigma)u^*_{l'}(\boldsymbol{q},\sigma) \mp e^{iq\cdot(x-x')}\sum_\sigma v_l(\boldsymbol{q},\sigma)v^*_{l'}(\boldsymbol{q},\sigma)\right), \tag{1.1.9}$$

and

$$-i\Delta^F_{ll'}(x,x') = \int \frac{d^3q}{(2\pi)^3 2\omega}\left(\theta(t-t')e^{-iq\cdot(x-x')}\sum_\sigma u_l(\boldsymbol{q},\sigma)u^*_{l'}(\boldsymbol{q},\sigma) \pm\theta(t'-t)e^{iq\cdot(x-x')}\sum_\sigma v_l(\boldsymbol{q},\sigma)v^*_{l'}(\boldsymbol{q},\sigma)\right). \tag{1.1.10}$$

Thus the essential quantities are the spin sums over the products of coefficient functions. Once these are found, it remains to combine the two terms in each of Δ and Δ^F in the form of four-dimensional Fourier transforms. While for Δ it can be done simply by introducing a mass shell delta function, it is a bit more complicated for Δ_F, involving derivatives of theta functions. The latter gives rise to non-covariant terms for fields of spin greater than $\frac{1}{2}$, which may be traced to the too-singular behaviour of commutators of such fields at the origin of the light-cone. However, these terms are local and so may be removed by adding a non-covariant piece in the Hamiltonian, making the resulting theory Lorentz covariant [1, 2].[1]

1.2 Scalar Field

The familiar Lagrangian (density) for a free real (hermitian) field

$$\mathscr{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1.2.1}$$

gives the dynamical (Euler–Lagrange) equation

$$(\Box + m^2)\phi(x) = 0. \tag{1.2.2}$$

[1] Another justification for ignoring the non-covariant terms can be obtained from the path integral formalism, where the propagator is obtained directly from the quadratic terms in the Lagrangian (and the vertices are read off from the interaction terms) [1].

The field $\phi(x)$ can then be expanded as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}\right), \quad \omega = +\sqrt{\boldsymbol{k}^2 + m^2}. \quad (1.2.3)$$

Here the coefficient functions are unity, there being no polarisation. So the field commutator (1.1.9) reduces to

$$\Delta(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega} \left(e^{-ik\cdot(x-x')} - e^{ik\cdot(x-x')}\right) \quad (1.2.4)$$

$$\equiv \Delta_+(x - x') - \Delta_+(x' - x) \quad (1.2.5)$$

where Δ_+ is the standard function

$$\Delta_+(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik\cdot(x-x')}. \quad (1.2.6)$$

On using the formula

$$\delta(k^2 - m^2) = \frac{1}{2\omega}\{\delta(k_0 - \omega) + \delta(k_0 + \omega)\}$$

it can be written as a four-dimensional Fourier transform

$$\Delta_+(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} 2\pi\theta(k_0)\delta(k^2 - m^2). \quad (1.2.7)$$

With this representation the commutator (1.2.5) takes the form

$$\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \rho_0(k) \quad (1.2.8)$$

where ρ_0 is the free spectral function given by

$$\rho_0(k) = 2\pi\epsilon(k_0)\delta(k^2 - m^2). \quad (1.2.9)$$

Here $\epsilon(k_0)$ is another step function, $\epsilon(k_0) \equiv \theta(k_0) - \theta(-k_0)$, which is $+1$ for $k_0 > 0$ and -1 for $k_0 < 0$.

The Feynman propagator formula (1.1.10) gives directly the spatial Fourier transform

$$-i\Delta_F(x, x') = \theta(t - t')\Delta_+(x - x') + \theta(t' - t)\Delta_+(x' - x)$$

$$= -i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \Delta_F(\omega, t - t') \quad (1.2.10)$$

where[2]

$$\Delta_F(\omega, t - t') = \frac{i}{2\omega}\{\theta(t - t')e^{-i\omega(t-t')} + \theta(t' - t)e^{i\omega(t-t')}\} \quad (1.2.11)$$

[2] To prove the second equality, we do the k_0 integral by closing the integration contour by a large semicircle: If $t > t'$, it is clockwise in the lower half-plane, when the integral picks up the residue at $k_0 = \omega - i\eta$; if $t < t'$, it is anticlockwise in the upper half-plane, when the integral picks up the residue at $k_0 = -\omega + i\eta$.

$$= -\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0(t-t')}}{\{k_0 + i\eta\epsilon(k_0)\}^2 - \omega^2}. \tag{1.2.12}$$

Here η is a positive infinitesimal quantity introduced to define the poles of the integrand.[3] As the integration is over all values of k_0, it is not restricted to mass-shell, $k_0^2 = \omega^2$. Inserting (1.2.12) into (1.2.10) we get the four-dimensional Fourier transform

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \Delta_F(k), \tag{1.2.13}$$

with $\Delta_F(k)$ given by

$$\Delta_F(k) = \frac{-1}{\{k_0 + i\eta\epsilon(k_0)\}^2 - \omega^2} = \frac{-1}{k^2 - m^2 + i\eta} \tag{1.2.14}$$

on noting that $2k_0\eta\epsilon(k_0)$ may be replaced simply by η. Notice that although both $\Delta(x)$ and $\Delta_F(x)$ are written as integrals over four momenta, only $\Delta_F(x)$ involves off-shell (virtual) momenta, originating from the integral (1.2.12). We can also put the free propagator (1.2.14) in the form of a spectral representation[4]:

$$\begin{aligned} \Delta_F(k) &= \int_{-\infty}^{\infty} dk_0' \frac{\{\delta(k_0' - \omega) - \delta(k_0' + \omega)\}/(2\omega)}{k_0' - k_0 - i\eta\epsilon(k_0)} \\ &= \int_{-\infty}^{+\infty} \frac{dk_0'}{2\pi} \frac{\rho_0(k_0', \boldsymbol{k})}{k_0' - k_0 - i\eta\epsilon(k_0)} \end{aligned} \tag{1.2.15}$$

in an apparently non-covariant form. As we shall see in Chapter 4, complete (interacting) thermal propagators will arise naturally in this form.

* * *

Theories with a multiplet of scalar fields are of much interest, as we shall see in the next chapter. Here we mention only the special case of a doublet, having the Lagrangian

$$\mathscr{L}(\phi_1, \phi_2) = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - \frac{m^2}{2}(\phi_1^2 + \phi_2^2). \tag{1.2.16}$$

For a compact notation we introduce a complex (non-hermitian) field defined by $\phi(x) = (\phi_1 + i\phi_2)/\sqrt{2}$, when the Lagrangian becomes

$$\mathscr{L}(\phi) = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi. \tag{1.2.17}$$

Clearly the complex field has the same expressions for the spectral function and propagator as the real field, if we define them following (1.1.9) and (1.1.10).

[3] Besides the time-ordered (Feynman) propagator, one can also define the retarded/advanced propagator by replacing the denominator in (1.2.12) with $(k_0 \pm i\eta)^2 - \omega^2$ respectively. Taking the retarded case and integrating over k_0 as before, we get zero for $t < t'$. Similarly for the advanced case, it is zero for $t > t'$.

[4] The argument of ϵ function can also be k_0', as it is only at the pole, $k_0' = k_0$, that its presence matters.

1.3 Dirac Field

The free Lagrangian for the Dirac field is

$$\mathscr{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{1.3.1}$$

where γ^μ are the so-called Dirac gamma matrices of dimension 4×4. It gives the Euler–Lagrange equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{1.3.2}$$

Applying $(i\gamma^\nu \partial_\nu + m)$ from the left and requiring the gamma matrices to satisfy the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{4\times 4}, \tag{1.3.3}$$

we get the Klein–Gordon equation

$$(\Box + m^2)\psi(x) = 0, \tag{1.3.4}$$

for each component of the Dirac field. As we discuss in Appendix A, the Dirac equation (1.3.2) itself may be interpreted as a supplementary condition, reducing the number of independent components of the field from four to two, which is the number of physical degrees of freedom of a spin-$\frac{1}{2}$ particle[3]. If an explicit choice of gamma matrices is necessary, we shall use the Weyl representation[5]:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \tag{1.3.5}$$

where $\mathbf{1}$ is the unit 2×2 matrix and σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3.6}$$

satisfying

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{1.3.7}$$

By inspection, we see that γ^0 is hermitian, while γ^i are antihermitian; the two results may be put together by writing $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Also $\gamma^{5\dagger} = \gamma^5$.

The field $\psi(x)$ can be expanded as $(\omega = \sqrt{\boldsymbol{p}^2 + m^2})$:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega} \sum_\sigma \left\{u(\boldsymbol{p},\sigma)e^{-ip\cdot x}a(\boldsymbol{p},\sigma) + v(\boldsymbol{p},\sigma)e^{ip\cdot x}b^\dagger(\boldsymbol{p},\sigma)\right\}. \tag{1.3.8}$$

Inserting this expansion in (1.3.2), the spinor coefficient functions u and v are found to satisfy

$$(\not{p} - m)u(\boldsymbol{p},\sigma) = 0, \qquad (\not{p} + m)v(\boldsymbol{p},\sigma) = 0. \tag{1.3.9}$$

[5] The other useful representation is that of Dirac, obtained by interchanging γ_0 and γ^5 in (1.3.5).

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We now evaluate the spin sums

$$M(p) \equiv \sum_\sigma u(\boldsymbol{p},\sigma)\overline{u}(\boldsymbol{p},\sigma), \qquad N(p) \equiv \sum_\sigma v(\boldsymbol{p},\sigma)\overline{v}(\boldsymbol{p},\sigma) \tag{1.3.10}$$

without solving for u and v. Recall that any 4×4 matrix may be expanded in terms of the 16 covariant matrices $\mathbf{1}$, γ^μ, $\sigma^{\mu\nu} \equiv \frac{i}{2}\{\gamma^\mu,\gamma^\nu\}$, $\gamma^\mu\gamma^5$ and γ^5. Here only one four-vector p^μ is available. So we may expand, for example, $M(p)$ as

$$M(p) = a\not{p} + b + c\not{p}\gamma^5 + d\gamma^5,$$

where a, b, c, d are constants. Apply $(\not{p} - m)$ on it from the left and also from the right. Subtracting one from the other, we get $c = d = 0$. Then any one of the equations gives $b = ma$. With the conventional normalisation of the field $\psi(x)$, we get both $M(p)$ and $N(p)$ in this way as

$$M(p) = \not{p} + m, \qquad N(p) = \not{p} - m, \tag{1.3.11}$$

satisfying $M^2 = 2mM$ and $N^2 = 2mN$. Note also that the two spin sums are related as $N(p) = -M(-p)$.

From (1.1.9) the field anticommutation relation is now given by

$$\begin{aligned}\{\psi(x),\overline{\psi}(x')\} &= \int \frac{d^3p}{(2\pi)^3 2\omega}\left\{(\not{p}+m)e^{-ip\cdot(x-x')} + (\not{p}-m)e^{ip\cdot(x-x')}\right\} \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}\sigma_0(p) \end{aligned}\tag{1.3.12}$$

where $\sigma_0(p)$ is the free spectral function

$$\sigma_0(p) = 2\pi\epsilon(p_0)(\not{p}+m)\delta(p^2 - m^2). \tag{1.3.13}$$

The Feynman propagator (1.1.10) becomes

$$\begin{aligned} -iS_F(x-x') &= \theta(t-t')\int \frac{d^3p}{(2\pi)^3 2\omega}(\not{p}+m)e^{-ip\cdot(x-x')} \\ &\quad -\theta(t'-t)\int \frac{d^3p}{(2\pi)^3 2\omega}(\not{p}-m)e^{ip\cdot(x-x')} \\ &= \theta(t-t')(i\not{\partial}+m)\Delta_+(x-x') + \theta(t'-t)(i\not{\partial}+m)\Delta_+(x'-x). \end{aligned}\tag{1.3.14}$$

If we could pull the theta functions past the (time) derivatives, the Dirac propagator would be related to the scalar one. But it produces an additional term $-i\gamma^0\delta(t-t')\Delta(x-x')$, which, however, turns out to be zero (Problem 1.1a). We thus get the Dirac propagator:

$$\begin{aligned} S_F(x-x') &= (i\not{\partial}+m)\Delta_F(x-x') \\ &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}S_F(p), \qquad S_F(p) = \frac{-(\not{p}+m)}{p^2 - m^2 + i\eta}. \end{aligned}\tag{1.3.15}$$

Like (1.2.15) in the scalar case, we have the spectral representation of the free Dirac propagator

$$S_F(p) = \int_{-\infty}^{+\infty} \frac{dp_0'}{2\pi} \frac{\sigma_0(p_0', \boldsymbol{p})}{p_0' - p_0 - i\eta\epsilon(p_0)}. \tag{1.3.16}$$

As we shall see at the end of Section 1.6, Feynman propagators for all higher-spin fields also admit such spectral representations.

1.4 Vector Field

We first derive the free Lagrangian for the vector field B^μ and the supplementary condition on it to describe a spin-one particle. The most general form may be written as [1, 4]:

$$\mathscr{L} = \frac{1}{2}\left\{a\,\partial_\mu B_\nu \partial^\mu B^\nu + b\,\partial_\mu B_\nu \partial^\nu B^\mu + c\,(\partial_\mu B^\mu)^2\right\} + \frac{m^2}{2} B_\mu B^\mu, \tag{1.4.1}$$

where a, b, c and m^2 are arbitrary constants. The Euler–Lagrange equation is

$$a\Box B^\mu + (b+c)\partial^\mu(\partial_\nu B^\nu) - m^2 B^\mu = 0. \tag{1.4.2}$$

Taking divergence it gives

$$(a+b+c)\Box\partial_\mu B^\mu - m^2 \partial_\mu B^\mu = 0, \tag{1.4.3}$$

which is the equation of motion for the scalar field $\partial_\mu B^\mu$ with $-m^2/(a+b+c)$ as the squared mass. Since we want here to describe only particles of spin one and not zero, we can avoid $\partial_\mu B^\mu$ as a propagating field by setting $a+b+c=0$, when we have $\partial_\mu B^\mu = 0$. Eliminating b, the Lagrangian (1.4.1) becomes

$$\mathscr{L} = \frac{a}{2}\partial_\mu B_\nu(\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{c}{2}(\partial_\mu B^\mu \partial_\nu B^\nu - \partial_\mu B_\nu \partial^\nu B^\mu) + \frac{m^2}{2} B_\mu B^\mu. \tag{1.4.4}$$

The second term turns out to be a total divergence,

$$\partial_\mu B^\mu \partial_\nu B^\nu - \partial_\mu B_\nu \partial^\nu B^\mu = \partial_\mu\{B_\nu(g^{\mu\nu}\partial_\rho B^\rho - \partial^\nu B^\mu)\}. \tag{1.4.5}$$

Omitting this term and absorbing a in the definition of B_μ and m, we get[6]

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}B_\mu B^\mu, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{1.4.6}$$

The dynamical equation for $B_\mu(x)$ is then simply

$$(\Box + m^2)B^\mu(x) = 0, \tag{1.4.7}$$

with the supplementary condition

$$\partial_\mu B^\mu = 0. \tag{1.4.8}$$

[6] We choose a to be negative so that the space-like components, which are the physical ones, can contribute negatively to the Lagrangian density. Then, for these components, the signs of different terms agree with those for the scalar case (1.2.1).

The condition (1.4.8) reduces the independent components of $B^\mu(x)$ from four to three, which is the number of degrees of freedom of a spin-one particle. (This is the subsidiary condition (A.3) of Appendix A.)

The vector field may again be expanded in terms of creation and annihilation operators,

$$B^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \sum_\sigma \{e^\mu(\boldsymbol{k},\sigma)e^{-ik\cdot x}a(\boldsymbol{k},\sigma) + e^{\mu *}(\boldsymbol{k},\sigma)e^{ik\cdot x}b^\dagger(\boldsymbol{k},\sigma)\} \tag{1.4.9}$$

when the supplementary condition gives

$$k_\mu e^\mu = 0. \tag{1.4.10}$$

The spin sum

$$E^{\mu\nu}(k) = \sum_\sigma e^\mu(\boldsymbol{k},\sigma)e^{\nu *}(\boldsymbol{k},\sigma) \tag{1.4.11}$$

must be a linear combination of the two available second rank tensors $g^{\mu\nu}$ and $k^\mu k^\nu$. The condition (1.4.10) fixes the relative coefficient and we choose the normalisation to write

$$E^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}, \tag{1.4.12}$$

satisfying $E^{\mu\nu}E_{\nu\lambda} = -E^\mu_\lambda$. Then the field commutator (1.1.9) becomes

$$\Delta^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} \rho_0^{\mu\nu}(k), \tag{1.4.13}$$

where the free spectral function is given by

$$\rho_0^{\mu\nu}(k) = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) 2\pi\epsilon(k^0)\delta(k^2 - m^2). \tag{1.4.14}$$

Also, the Feynman propagator (1.1.10) is given by

$$\begin{aligned} -i\Delta_F^{\mu\nu}(x-x') &= \theta(t-t')\left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(x-x') \\ &\quad + \theta(t'-t)\left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_+(x'-x). \end{aligned} \tag{1.4.15}$$

In pulling the theta functions past the tensors, the time derivatives may produce additional terms. As we saw in the case of the Dirac propagator, the first derivative does not produce any, but the second derivative produces one. Using the properties of $\Delta(x-x')$ in Problems 1.1(a) and (b), one gets

$$\begin{aligned} &\theta(t-t')\partial_t^2\Delta_+(x-x') + \theta(t'-t)\partial_t^2\Delta_+(x'-x) \\ &= -i\partial_t^2\Delta_F(x-x') + i\delta^4(x-x'). \end{aligned} \tag{1.4.16}$$

The propagator (1.4.15) then becomes

$$\Delta_F^{\mu\nu}(x-x') = \left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right)\Delta_F(x-x') + \frac{1}{m^2}\delta_0^\mu\delta_0^\nu\delta^4(x-x'). \tag{1.4.17}$$

Taking a Fourier transform, one gets

$$\Delta_F^{\mu\nu}(k) = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)\frac{-1}{k^2-m^2+i\epsilon} + \frac{1}{m^2}\delta_0^\mu\delta_0^\nu. \tag{1.4.18}$$

As already mentioned, we may drop the non-covariant local term to write it simply as

$$\Delta_F^{\mu\nu}(k) = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)\frac{-1}{k^2-m^2+i\epsilon}. \tag{1.4.19}$$

* * *

For a massless vector field $A_\mu(x)$, the propagator (1.4.19) may still be used if the interaction is linear in $A_\mu(x)$, $\mathscr{L}_{int} \sim J^\mu(x)A_\mu(x)$ and the current $J_\mu(x)$ is conserved, ($\partial^\mu J_\mu = 0$), as in electrodynamics. Then in matrix elements, where the photon propagator is sandwiched between currents, the $k^\mu k^\nu/m^2$ term vanishes. So it amounts to taking

$$\Delta^{\mu\nu}(k)|_{m=0} = \frac{g^{\mu\nu}}{k^2+i\epsilon}. \tag{1.4.20}$$

1.5 Rarita–Schwinger Field

A spin $\frac{3}{2}$ particle can be described by a Rarita–Schwinger field ψ_A^μ, a vector-spinor having $4\times 4 = 16$ components [5]. Following Appendix A we reduce the number of independent components to four, needed to describe the spin components. Suppressing the Dirac index as before, we impose

$$\gamma_\mu\psi^\mu(x) = 0 \tag{1.5.1}$$

to take away four components and then the Dirac equation

$$(i\partial\!\!\!/ - m)\psi^\mu(x) = 0 \tag{1.5.2}$$

to remove another eight components, retaining just four of them. A different supplementary condition

$$\partial_\mu\psi^\mu(x) = 0 \tag{1.5.3}$$

follows from the first two conditions[7].

We shall not try to write the general form of the Lagrangian for this field, which is somewhat involved [6]. As before we expand $\psi^\mu(x)$ in terms of positive and negative frequency modes,

[7] To get it, multiply (1.5.2) by γ_μ from the left and use anticommutation relation (1.3.3) of gamma matrices and the other supplementary condition (1.5.1).