

A Primer on Fourier Analysis for the Geosciences

Time-series analysis is used to identify and quantify periodic features in datasets and has many applications across the geosciences, from analysing weather data, to solid-Earth geophysical modelling. This intuitive introduction provides a practical 'how-to' guide to basic Fourier theory, with a particular focus on Earth system applications. The book starts with a discussion of statistical correlation, before introducing Fourier series and building to the Fast Fourier Transform (FFT) and related periodogram techniques. The theory is illustrated with numerous worked examples using R datasets, from Milankovitch orbital-forcing cycles to tidal harmonics and exoplanet orbital periods. These examples highlight the key concepts and encourage readers to investigate more advanced time-series techniques. It concludes with a consideration of statistical effect-size and significance. This useful book is ideal for graduate students and researchers in the Earth system sciences who are looking for an accessible introduction to time-series analysis.

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Robin Crockett

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For The Elf, The Moth and The Beagle.

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Preface

This book evolved from the Fast Fourier Transform (FFT) and time-series short-courses I have given at the European Geosciences Union (EGU) General Assemblies over several years. Those short-courses were, in turn, an evolution from *ad hoc* sets of postgraduate tutorial materials to something more organised and integrated, *i.e.* from one-to-one response-to-questions delivery to one-to-many questions-anticipated delivery.

I did not know what to expect when I gave the first EGU short-course in 2009 – would anyone even turn up? Well, many more people attended than the allocated room could sensibly accommodate – and, needless to say, I did not anticipate all the questions. However, its success led to an invitation to give a similar short-course the following year – which again more than filled the bigger room allocated. Since then the course has rolled forward from year to year, each version maintaining the core FFT coverage but with varying details in response to suggestions and feedback from the previous year.

This book is a primer and so, by definition, cannot cover everything. It cannot even cover every type of geoscience time-series but it aims to give early-stage researchers, including research students, in the geosciences – and other sciences – a basic but robust grasp of the essential properties of the FFT in the context of time-series analysis. It focuses on that one specific task and, noting that it is intended for non-specialists to fill in the details that more advanced books on Fourier theory tend to pass over, endeavours to do this by presenting intuitive and illustrative theory and explanations rather than more abstract ones. As well as presenting core theory and practice for those wishing to use the FFT for time-series analysis, it goes a little deeper into the theory to give early-stage researchers sufficient depth and awareness of the limitations in order to critically evaluate other people's application and interpretation of the FFT.

I hope you find it useful both of itself and as a pointer and introduction to more advanced texts.

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Going further back, I would like to thank Bruce Malamud who first invited me to give a short-course at the 2009 European Geosciences Union General Assembly. I would also like to thank Phil Picton and Frédéric Perrier who said ‘just do it’ and gave me valuable support and encouragement throughout. And ... everyone who has developed and disseminated Fourier theory before me.