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Introduction

1.1 Introduction to Geometric Rigidity

Our story begins with the Bieberbach theorems about the structure of compact flat manifolds (i.e. compact Riemannian manifolds whose sectional curvatures are everywhere 0, i.e. that are locally isometric to $\mathbb{R}^n$). The universal cover of such a manifold, M, is Euclidean space, and therefore its fundamental group π is a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^n)$. There is a (split) exact sequence

$$1 \to \mathbb{R}^n \to \mathrm{Iso}(\mathbb{R}^n) \to \mathrm{O}(n) \to 1$$

so that π has a rotational part, and a translation subgroup. (Thus $\mathrm{Iso}(\mathbb{R}^n)$ is a semidirect product of the linear = orthogonal group, and the group of translations, where the former acts on the latter in the obvious way.)

Bieberbach showed that the rotational part of π is always finite, so that π has a subgroup of finite index that is pure translation, and simple considerations then guarantee that this is rank n, i.e. that M is finitely covered by a torus, i.e. by $\mathbb{R}^n/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^n$.

We shall first assume that this is a 1-fold cover for simplicity:[1] the structure of the manifold M we started with is then understood as a structure on a torus, and by an analysis of its isometries.

The space of tori, though, is very interesting and quite nontrivial already. (Indeed the $n = 2$ case gives rise to the beautiful theory of modular forms (Serre, (1973).) Let us normalize by demanding that $\mathrm{vol}(M) = 1$, and furthermore let us pick the isomorphism $\Lambda \to \mathbb{Z}^n$ (which is tantamount to giving a homotopy equivalence $M \to \mathbb{T}^n$). There is a unique linear map in $\mathrm{GL}_n(\mathbb{R})$ taking $\Lambda \to \mathbb{Z}^n$. Notice that the translation group is conjugate to the standard action (as a group action of $\mathbb{Z}^n$) iff (if and only if) this matrix is orthogonal. Thus, the space of

[1] Although this is but one of a superexponentially growing number of possibilities as n increases.

"polarized flat tori of volume 1" is the same as $SL_n(\mathbb{R})/SO(n)$, a contractible manifold – e.g. by the Gram–Schmidt process.[2]

At this point, we can pick up the theory for general flat manifolds if we want: the finite holonomy group (the group of rotations we ignored before) acts on the space of flat tori, and whose fixed point set is the space of flat structures on the given manifold (with volume equal to 1/#holonomy). The fixed set of a compact group acting on a complete simply connected non-positively curved manifold is another such space, by a theorem of Hadamard provided it is nonempty and connected. It is nonempty (in general, this is Cartan's fixed-point theorem: a fixed point can be given as the unique "median" of any orbit – the point which makes the largest distance to any point of the orbit finite) in our case, because we assumed there was a flat manifold, and connected, because a geodesic connecting two fixed points to each other would be fixed and therefore lie in the fixed set. Anyway, we then see that there is a unique such manifold as a smooth manifold, and that any two are conjugate in the affine group.

Mostow (1968) showed in a celebrated paper that for constant negative curvature manifolds, the rigidity is much stronger. Perhaps the first hint of this comes from the Gauss–Bonnet theorem: In this case it says that:

Proposition 1.1 *If M is a closed manifold*[3] *of constant curvature* -1, *i.e. if M is a closed hyperbolic manifold of even dimension, then*

$$\chi(M^{2n}) = 2(-1)^n \operatorname{vol}(M)/\omega_{2n},$$

where ω_{2n} is the volume of the sphere (of radius 1*).*

To foreshadow other developments, we note that if $\operatorname{vol}(M) < \infty$, then M has finite topological type (i.e. is the interior of a compact manifold with boundary) so that both sides of the equation make sense, and in fact the equation holds.

As a consequence of the Gauss–Bonnet theorem, we see that in the hyperbolic case, unlike the flat case, the fundamental group determines the volume.[4] Perhaps even more straightforwardly, flat manifolds have a nonrigidity because of homotheties, but hyperbolic manifolds have a scale because of their nonvanishing curvature.

Mostow's theorem then gives what seems like the ultimate strengthening

[2] In the spirit of later developments, we should say that $SL_n(\mathbb{R})/SO(n)$ is a complete simply connected manifold of non-positive curvature – as is any semisimple Lie group modulo its maximal compact subgroup – and is thus, by Hadamard's theorem, diffeomorphic to Euclidean space.

[3] Recall that a closed manifold is a compact manifold without boundary.

[4] At least in even dimensions. Mostow rigidity implies that this is true in all dimensions; a cohomological explanation for this is provided by Gromov's theory of bounded cohomology (Gromov, 1982).

of this line of thought. The contractible manifold occurring in the flat case degenerates (if the dimension is greater than >2) to a point!

Theorem 1.2 *Suppose that M and M' are closed hyperbolic manifolds of dimension $d > 2$, then any isomorphism $h\colon \pi_1(M) \to \pi_1(M')$ is induced by a unique isometry between M and M'.*

As a minor point, strictly speaking, an induced map on fundamental groups requires the map to preserve base points, but the isometry will almost surely not (as it's unique, it either does or does not). Consequently, we should actually assume that one has a conjugacy class of homomorphisms of the fundamental group, or use groupoids, or some similar device.

We note that this is not true in dimension 2; for a surface of genus g, the space of marked[5] hyperbolic structures is called Teichmuller space, and is topologically $\mathbb{R}^{6g-6}$.

Mostow's theorem is a beautiful and perhaps initially surprising result. However, it can feel a bit sterile if one doesn't know examples of hyperbolic manifolds and indeed it is not so easy to construct hyperbolic manifolds in dimension >2 (in dimension 2 they can be built easily using tessellations of the hyperbolic plane). Even after knowing some constructions, how are you going to find two not obviously isometric hyperbolic manifolds that have isomorphic fundamental groups?

However, the uniqueness statement in Mostow's theorem gives us quite nontrivial information even when $M = M'$. Any self-isomorphism of π must be realized by a self-isometry, giving the following conclusion:

Corollary 1.3 *If π is the fundamental group of a compact hyperbolic manifold M, then $\mathrm{Iso}(M) \cong \mathrm{Out}(\pi)$, where $\mathrm{Iso}(M)$ is the isometry group of M, and $\mathrm{Out}(\pi)$ is the group of outer automorphisms of π: it is the quotient of the automorphisms $\mathrm{Aut}(\pi)$ by $\mathrm{Inn}(\pi)$, the normal subgroup of inner automorphisms of π.*

$\mathrm{Out}(\pi)$ is the set of components of the self-homotopy equivalences of M to itself: it is not $\mathrm{Aut}(\pi)$ because we do not insist that maps and homotopies preserve base points.

The isometry group of a compact manifold is always a compact Lie group (Myers–Steenrod), so we learn that, in the hyperbolic case, this group is always finite. Then we then deduce the purely algebraic fact that $\mathrm{Out}(\pi)$ must be finite – if the dimension of the hyperbolic manifold >2.

In dimension 2, the first conclusion holds (as we will discuss later), but the second does not. $\mathrm{Out}(\pi)$ is the celebrated mapping class group, an object of

[5] That is, ones where we are given an identification of the fundamental group or, equivalently, a homotopy class of a map to a standard surface.

fundamental importance in low-dimensional topology and in algebraic geometry. Elements of infinite order in Out(π) can never be realized by isometries of a compact Riemannian manifold.

The conclusions of Mostow's theorem can be greatly generalized. First of all, hyperbolic space can be generalized to be any locally symmetric manifold with no Euclidean factors and no hyperbolic plane factors: in other words, as Mostow showed in subsequent work, it applies to G/K, if G is a semisimple Lie group (i.e. a Lie group with no connected normal solvable subgroups) and K its maximal compact subgroup. We will discuss these in much greater length in Chapter 2.

In addition, according to Prasad (1973), all of these rigidity theorems hold for noncompact finite volume hyperbolic manifolds (and locally symmetric manifolds).

Amazingly enough, there are many additional extensions of these theorems, not thought of as uniqueness theorems *per se*. We will discuss some of the important work of Margulis on (the aptly called) superrigidity in Chapter 2.

1.2 The Borel Conjecture

The striking results of §1.1 show that for various "geometric structures" (let's say that this means a given choice of a local model for germ neighborhoods of points), the space of given marked structured manifolds is either a point, or the algebraic topologist's "point": a contractible space.[6]

Although a contractible space isn't as good as a point, for some purposes it's quite good. For example, that it is connected is already a type of uniqueness statement. In the situation where one has a structure on this space with non-positive curvature, one can geometrically make conclusions that are stronger than follow from the algebraic topology alone. For instance, the non-positive curvature on the space of flat tori enables one to prove Bieberbach's theorem that any torsion-free group that is virtually free abelian (of rank k) is the fundamental group of a compact aspherical manifold (of dimension k). (Exercise or see the footnotes.)

Borel suggested that the topological conclusion that the hyperbolic manifolds were homeomorphic[7] could be traced to a purely topological hypothesis:

[6] There should be a kind of mathematician for whom a point is a non-positively curved space – someone informed by both algebraic and geometric intuitions.

[7] Borel actually made his conjecture on the basis of an earlier result of Mostow (1954) on solvmanifolds, where the conclusion was "isomorphic," i.e. diffeomorphic. Borel expressed relief that he hadn't conjectured diffeomorphic in light of this result.

Conjecture 1.4 *If $h \colon M' \to M$ is a homotopy equivalence between closed aspherical manifolds, then h is homotopic to a homeomorphism.*

Recall that a space is aspherical if its universal cover is contractible. It is a $K(\pi, 1)$ in the language of the algebraic topologists, meaning its homotopy groups π_i vanish for $i > 1$. This can be tested by checking whether the universal cover has vanishing reduced integral homology (by the Hurewicz isomorphism theorem). A homotopy (class of) equivalence(s) between aspherical spaces is essentially the same thing as a (conjugacy class of) isomorphisms between their fundamental groups.

If M is non-positively curved or of the form $K\backslash G/\Gamma$ (where G is a real Lie group and K its maximal compact), then it satisfies the hypothesis of the Borel conjecture. In these cases, the conjecture is an astounding theorem of Farrell and Jones.[8]

One can also try to reverse this mode of thought, and ask whether the moduli space of non-positively curved structures on a closed topological aspherical manifold is contractible (if it is nonempty!). Farrell and Jones have shown that the answer to this is negative as well: the space isn't even connected. But, I am running ahead of the story.

Borel is suggesting here that aspherical is the topological analogue of "locally symmetric of noncompact type" or of "non-positively curved." In Chapter 2 we will discuss various constructions of aspherical manifolds – although in Borel's time there were no examples that were very far away from the lattice setting.

Of course, in the topological setting, one cannot expect the homeomorphism to be unique. However, it might seem reasonable to believe that the space of homeomorphisms is contractible, i.e. the analogue of a point. Unfortunately, this is not true and we will later discuss the reason for this; it is an indirect consequence of the conjecture that there is a type of uniqueness: uniqueness up to pseudo-isotopy.

Definition 1.5 Two homeomorphisms $f, g \colon M \to N$ are pseudo-isotopic if there is a homeomorphism $M \times [0,1] \to N \times [0,1]$ that restricts to $f \cup g$ on the boundary $M \times \{0,1\} \to N \times \{0,1\}$.

For high-dimensional closed manifolds, one knows due to the work of Cerf and of Hatcher and Wagoner (see Hatcher and Wagoner, 1973) that pseudo-isotopies between homeomorphisms are isotopic to isotopies iff the manifolds are simply connected. This work shows that always there's typically an infinite number of isotopy classes of homeomorphisms in the given homotopy class.

8 The important point being that M' is *not* assumed to be a space of this sort (for then, the relevant result is part of differential geometric rigidity). We will explain some of the ideas of this result in Chapter 8.

The reasonable optimist might therefore choose to append "unique up to pseudo-isotopy" to the statement of the Borel conjecture. As we will discuss in Chapter 3, this both follows from the Borel conjecture in general, and is part of the "correct" natural extension to manifolds with boundary.

Uniqueness up to pseudo-isotopy is not as strong as uniqueness, and it will need some study. If one had uniqueness in families, one could immediately learn things about bundles. The weaker type of uniqueness has implication for "block bundles"[9] and has more relevance to the topological category than the bundle result would have (in other words, this is a feature, not a bug).

As we noted in the geometric setting, uniqueness would also immediately have implications regarding the symmetries of aspherical manifolds. Borel himself proved some of these, and we will discuss them in Chapter 7. For example, if M is an aspherical manifold whose fundamental group has trivial center, then the only connected compact Lie group that can act continuously on it is trivial.[10] We saw that it implied that any finite subgroup of $\mathrm{Out}(\pi)$ was realized by a group action – at least when π is centerless;[11] this statement is called the Nielsen realization problem.[12]

It also would imply certain uniqueness statements about group actions – or if you like, it would imply "equivariant Borel conjectures." We will see, by contrast (in Chapter 6), that these conjectures are false – for several different reasons.

Another variant of the Borel conjecture goes like this: Given a group π, the Borel conjecture asserts the uniqueness of the aspherical topological manifold whose fundamental group is π. Shouldn't there be an existence theorem to go with such a uniqueness one? Wall (1979) has conjectured that the correct condition is that π should satisfy Poincaré duality.[13] We will discuss some of the evidence for Wall's conjecture – most comes from the Borel conjecture –

[9] And even more to "approximate fibrations," which it would surely be taking us far afield to introduce at this point. Let us leave it as saying that if one tried to extend the Borel philosophy to some singular settings, and took seriously the idea that one is looking for topologically invariant notions rather than modeling closely the topological analogue of the smooth category, then one would be led to "pseudo"s.

It is worth noting that Mostow's work on hyperbolic manifolds is based on extending the map of universal covers to certain ideal ∂s. These extensions, as is critical to Mostow's work, are naturally continuous and not smooth. These ideas of Mostow from the late 1960s are fundamental to almost all of the work on the Borel conjecture since the early 1980s.

[10] Equivalently, every continuous circle action on M is trivial.

[11] When π is not centerless, the isometries tend not to be unique, and the realization is false for certain nilmanifolds, an example of Raymond and Scott (1977).

[12] The original Nielsen problem was for surfaces and was proven true first by Kerckhoff (1983) using geometrical properties of Teichmuller space. By now, there are a number of proofs.

[13] For a group to satisfy Poincaré duality it means that its $K(\pi, 1)$ satisfies Poincaré duality. In Wall's conjecture, one means Poincaré duality with arbitrary coefficient systems, to the same extent that one has such Poincaré duality for manifolds. This is equivalent to there being a

and we'll also discuss variants of Wall's conjecture where one weakens the type of Poincaré duality the group satisfies.

Yet another way of thinking about Borel's philosophy is the following. If knowing the group means knowing the manifold, then every topological property of manifolds has to be reflected in its fundamental group. Thus one can conjecture that an aspherical manifold is a nontrivial product iff its fundamental group is.[14] Similarly one can hope that a manifold "fibers" over another if there is a suitable exact sequence of groups. We will discuss these kinds of problems later.

If one were a wild optimist,[15] one could easily go very far and conjecture that many properties of the model manifolds hold for all aspherical manifolds, such as that their universal covers are Euclidean space or that their fundamental groups have solvable word problems. We will see in Chapter 2 that these are false.

It is not known whether their Euler characteristic has the same sign as the symmetric spaces of the same dimension have, i.e. whether $(-1)^n\chi(M^{2n}) \geq 0$, for closed aspherical manifolds. (This is sometimes called the Hopf conjecture, although Hopf only asked it for negatively curved manifolds.[16])

Finally, the Borel conjecture begets many others in the following indirect way: It implies that any method one would try to disprove it must fail. Thus any invariant of manifolds, defined by any method at all, no matter how clever or indirect, should be a homotopy invariant for aspherical manifolds. This means that the fundamental group must somehow catch lots of subtle geometry. Examples of this include the tangent bundle and various types of spectral invariants, but, in principle, one can consider any topological invariant at all.[17]

When studying this in detail, one is often led to problems that seemingly have nothing to do with aspherical manifolds. In Chapter 4 we will follow this road towards the Novikov conjecture, which in its analytic form has strong differential geometric implications – well beyond aspherical manifolds. In this form, the conjecture also develops analogues in quadratic form theory and in algebraic K-theory.

chain homotopy equivalence (with the usual dimensional shift) between the $\mathbb{Z}\pi$ chain complexes of singular chains on the universal cover and its dual.

14 This can be compared with a theorem of Lawson and Yau (1972) for non-positively curved manifolds.

15 Something that one would not ordinarily say of Borel.

16 Recently, Avramidi (2014) gave some very striking evidence for the failure of this conjecture.

17 An example of this includes simplicial volume *à la* Gromov (1982), which provides a homological explanation for the volume of certain locally symmetric manifolds. I mention this here because, unfortunately, it does not play a large role in what follows.

1.3 Notes

A good grounding in differential geometry is very helpful. For our purposes, Cheeger and Ebin (2008) is probably the best source. Milnor (1963), which is a rapid course in Riemannian geometry in *Morse Theory*, is adequate for most purposes in this book.

There are now a lot of approaches to Mostow rigidity and it has many extensions and generalizations. The original sources are Mostow (1968, 1973). I highly recommend the survey of Gromov and Lawson (1991). Probably the “easiest” proof (although one that is rather atypical) is Gromov’s based on the ideas of “bounded cohomology.” An excellent exposition of this can be found in Munkholm (1979). Zimmer (1984) gives a clear treatment of Margulis’s superrigidity theorem.

The discussion here of the Borel conjecture is not the most direct or efficient. However, the equivalent statement that “the structure set of an aspherical manifold vanishes” reduces all of one’s study to proving that some group is 0. This seems (to me) rather depressing. We prefer the point of view that the subject deals with actual *examples* and contains surprises. It makes it feel like one is actually studying *something* (Figure 1.1).

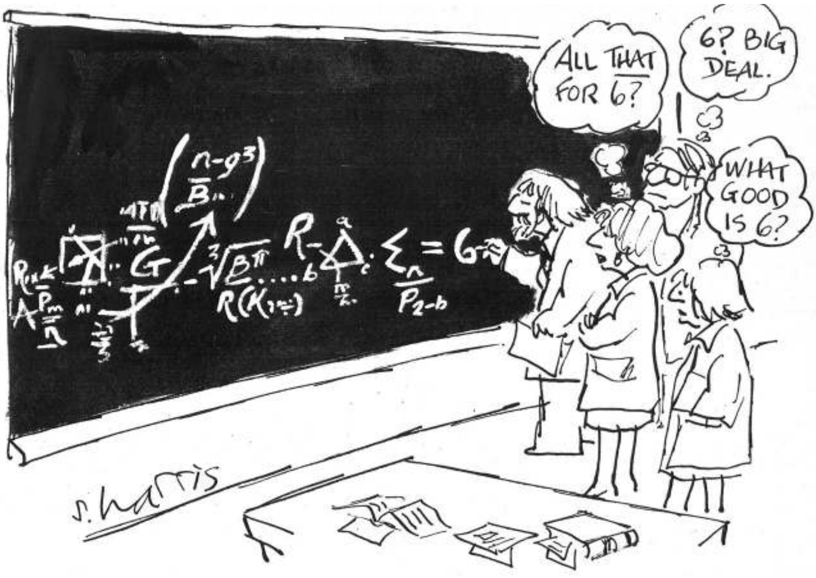

Figure 1.1 Six cartoon. © Sidney Harris, reproduced with permission. (`http://sciencecartoonsplus.com/`).

More seriously, the variants we consider shed light on some subtleties and possible approaches to the conjecture, and are, I think, natural questions that one would want to address for the same reason as one would want to know the truth of the Borel conjecture.[18]

And, finally, I hope that when the problem is ultimately solved, the spirit of the problem – as expanded on here – will continue to inspire future generations of mathematicians.

[18] So we prefer a *Comedy of Errors* to *Much Ado About Nothing*.