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STOCHASTIC ANALYSIS

Itô and Malliavin Calculus in Tandem

Thanks to the driving forces of the Itô calculus and the Malliavin calculus, stochastic analysis has expanded into numerous fields including partial differential equations, physics, and mathematical finance. This book is a compact, graduate-level text that develops the two calculi in tandem, laying out a balanced toolbox for researchers and students in mathematics and mathematical finance. The book explores foundations and applications of the two calculi, including stochastic integrals and stochastic differential equations, and the distribution theory on Wiener space developed by the Japanese school of probability. Uniquely, the book then delves into the possibilities that arise by using the two flavors of calculus together. Taking a distinctive, path-space-oriented approach, this book crystalizes modern day stochastic analysis into a single volume.

Hiroyuki Matsumoto is Professor of Mathematics at Aoyama Gakuin University. He graduated from Kyoto University in 1982 and received his Doctor of Science degree from Osaka University in 1989. His research focuses on stochastic analysis and its applications to spectral analysis of Schrödinger operations and Selberg's trace formula, and he has published several books in Japanese, including *Stochastic Calculus* and *Introduction to Probability and Statistics*. He is a member of the Mathematical Society of Japan and an editor of the *MSJ Memoirs*.

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Stochastic Analysis

Itô and Malliavin Calculus in Tandem

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Aoyama Gakuin University, Japan

Setsuo Taniguchi
Kyushu University, Japan

Translated and adapted from the Japanese edition

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Preface

The aim of this book is to introduce stochastic analysis, keeping in mind the viewpoint of path space. The area covered by stochastic analysis is very wide, and we focus on the topics related to Brownian motions, especially the Itô calculus and the Malliavin calculus. As is widely known, a stochastic process is a mathematical model to describe a randomly developing phenomenon. Many continuous stochastic processes are driven by Brownian motions, while basic discontinuous ones are related to Poisson point processes.

The Itô calculus, named after K. Itô who introduced the calculus in 1942, is typified by stochastic integrals, Itô's formula, and stochastic differential equations. While Itô investigated those topics in terms of Brownian motions, they are now studied in the extended framework of martingales. One of the important applications of the calculus is a construction of diffusion processes through stochastic differential equations. The Malliavin calculus was introduced by P. Malliavin in the latter half of the 1970s and developed by many researchers. As he originally called it "a stochastic calculus of variation", it is exactly a differential calculation on a path space. It opened a way to take a purely probabilistic approach to transition densities of diffusion processes, which are fundamental objects in theory and are applied to many fields in mathematics and physics.

We made the book self-contained as much as possible. Several preliminary facts in analysis and probability theory are gathered in the Appendix. Moreover, a lot of examples are presented to help the reader to easily understand the assertions. This book is organized as follows. Chapter 1 starts with fundamental facts on stochastic processes. In particular, Brownian motions and martingales are introduced and basic properties associated with them are given. In the last three sections, investigations of path space type are made; the Cameron–Martin theorem, Schilder's theorem and an analogy with path integrals are presented.

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Chapter 2 introduces stochastic integrals and Itô's formula, an associated chain rule. Although Itô originally discussed them with respect to Brownian motions, we formulate them with respect to martingales in the recent manner due to J. L. Doob, H. Kunita and S. Watanabe. Moreover, several facts on continuous martingales are discussed: for example, representations of them by time changes and those via stochastic integrals with respect to Brownian motions.

Chapter 3 presents several properties of Brownian motion. As direct applications of Itô's formula, problems in the theory of partial differential equations, like heat equations and Dirichlet problems, are studied. Although the Laplacian is only dealt with in this chapter, after reading Chapters 4 and 5, the reader will be easily convinced that the results in this chapter can be extended to second order differential operators on Euclidean spaces and Laplace–Beltrami operators on Riemannian manifolds.

Chapters 4 and 5 form the main portion of this book. Chapter 4 introduces stochastic differential equations and presents their properties and applications. Stochastic differential equations enable us to construct diffusion processes in a purely probabilistic manner. Namely, diffusion processes are realized as measures on a path space via solutions of stochastic differential equations. This is different from the analytical method by A. Kolmogorov, which uses the fundamental solution of the associated heat equation. It is also seen in the chapter that stochastic differential equations determine stochastic flows as ordinary differential equations. The flow property will be used in the next chapter.

The Malliavin calculus is developed in Chapter 5. The distribution theory on the Wiener space, which was structured by the Japanese school led by S. Watanabe, S. Kusuoka, and I. Shigekawa, is introduced. Moreover, the integration by parts formula and the change of variable formula on the Wiener space are presented. In the last two sections, the latter formula is applied to computing Laplace transforms of quadratic Wiener functionals.

Chapter 6 is a brief introduction to mathematical finance. In this chapter, we focus on the Black–Scholes model, the simplest model in mathematical finance. The existence and uniqueness of an equivalent martingale measure is shown and a pricing formula of European contingent claims is given. Moreover, as an application of the Malliavin calculus, we show ways to compute hedging portfolios and the Greeks, indices to measure sensitivity of prices with respect to parameters like initial price and volatilities.

Stochastic analysis is the analysis on path spaces, and it is deeply related to Feynman path integrals. It was M. Kac who gained an insight into this close relationship and achieved a lot of results. His achievements exerted great influence on not only probability theory but also other fields of mathematics.

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Chapter 7 is intended to present results corresponding to such close relationship. It starts with an introduction of a Wiener space analog of the representation of a propagator by action integrals of classical paths, which was due to the physicist Van Vleck playing an active role in the early period of quantum mechanics. Next, applications of stochastic analysis to studies of eigenvalues of Schrödinger operators and Selberg's trace formula are presented. In these applications, probabilistic representations of a heat kernel with the aid of the Malliavin calculus provide a clear route to the results.

This book is based on the Japanese one *Kakuritsu Kaiseki* published by Baifukan in 2013. In this book, a section discussing the close conjunction of the Malliavin calculus and the rough path theory and a chapter on mathematical finance are newly added. During the writing of the Japanese book and this one, we have received much benefit from several representative monographs and books on stochastic analysis; these include:

- R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth, 1984;
- N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edn., North Holland/Kodansha, 1989;
- I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd edn., Springer-Verlag, 1991;
- D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 3rd edn., Springer-Verlag, 1999.

As for the theory of martingales, we also gained some benefit from

- H. Kunita, *Estimation of Stochastic Processes* (in Japanese), Sangyou Tosho, 1976.

This book did not come into existence without the Japanese one. We deeply thank Professor Nobuyuki Ikeda, our supervisor, who recommended our writing the Japanese book. In writing the Japanese version, we received kind assistance and help from several people, whom we gratefully acknowledge. Professor Yoichiro Takahashi was on the editorial board of the series where our book appeared and encouraged our writing. He and Professors Masanori Hino, Yu Hariya, and Koji Yano read through the draft and gave us stimulating comments and kind help.

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Frequently Used Notation

$\mathbb{N} \equiv \{1, 2, 3, \ldots\}$, $\mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$

$a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$

$a^+ = \max\{a, 0\}$, $a^- = -\min\{a, 0\} = \max\{-a, 0\}$

$[a]$: the largest integer less than or equal to $a \in \mathbb{R}$

$[s]_n = 2^{-n}[2^n s] \in \{k2^{-n}\}_{k=0}^{\infty}$ $(s \geqq 0,\ n \in \mathbb{N})$

$\mathrm{sgn}(x) = -1\ (x \leqq 0),\ = 1\ (x > 0)$

$\mathbf{1}_A$: the indicator function of a set A

$\sigma(\mathscr{C})$: the σ-field generated by a family $\mathscr{C}$ of subsets

$\sigma(X_1, \ldots, X_n)$: the σ-field generated by the random variables $X_1, \ldots, X_n$

$\mathscr{B}(S)$: the σ-field generated by open subsets of a topological space S

$C_0(S)$: the space of continuous functions on S with compact supports

$C^n(\mathbb{R}^d)$: the space of C^n-class functions on $\mathbb{R}^d$

$C_0^\infty(\mathbb{R}^d)$: the space of C^∞-functions on $\mathbb{R}^d$ with compact supports

$\mathscr{S}(\mathbb{R}^d)$: the space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$

$\mathscr{S}'(\mathbb{R}^d)$: the dual space of $\mathscr{S}(\mathbb{R}^d)$, the space of tempered distributions

Δ : the Laplacian on $\mathbb{R}^d$, $\sum_{i=1}^d \left(\frac{\partial}{\partial x^i}\right)^2$

$g(t, x, y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|y-x|^2}{2t}}$ $(t > 0,\ x, y \in \mathbb{R}^d)$: the Gauss kernel

$\mathbf{W} = \mathbf{W}^d = C([0, \infty) \to \mathbb{R}^d)$: the space of $\mathbb{R}^d$-valued continuous functions

$W = W^d = \{w \in \mathbf{W}^d;\ w(0) = 0\}$: the Wiener space

$W_T = W_T^d = C([0, T] \to \mathbb{R}^d)$: the Wiener space on $[0, T]$

$H_T = H_T^d$: the Cameron–Martin space of W_T^d

$\langle X \rangle$: the quadratic variation process of a semimartingale X

$\mathscr{M}^2$: the set of square-integrable martingales

$\mathscr{M}^2_{\mathrm{loc}}$: the set of square-integrable local martingales

$\mathscr{M}^2_{\mathrm{c,loc}}$: the set of square-integrable continuous local martingales

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