1

Fundamentals of Continuous Stochastic Processes

In this chapter fundamentals of continuous stochastic processes are mentioned, taking into account their applications to stochastic analysis.

1.1 Stochastic Processes

Let Ω be a set.

Definition 1.1.1 A family $\mathscr{F}$ of subsets of Ω is said to be a ***σ*-field** if

(i) $\Omega, \emptyset \in \mathscr{F}$,
(ii) if $A \in \mathscr{F}$, then $A^c := \{\omega \in \Omega; \omega \notin A\} \in \mathscr{F}$,
(iii) if $A_i \in \mathscr{F}$ $(i = 1, 2, \ldots)$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$.

The pair $(\Omega, \mathscr{F})$ is called a **measurable space**.

Definition 1.1.2 Let $(\Omega, \mathscr{F})$ be a measurable space. A set function $\mathbf{P} : \mathscr{F} \ni A \mapsto \mathbf{P}(A) \geqq 0$ is said to be a **probability measure** if

(i) $0 \leqq \mathbf{P}(A) \leqq 1$ for all $A \in \mathscr{F}$,
(ii) $\mathbf{P}(\Omega) = 1$,
(iii) for mutually disjoint subsets $A_i \in \mathscr{F}$ $(i = 1, 2, \ldots)$,

$$\mathbf{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbf{P}(A_i).$$

The triplet $(\Omega, \mathscr{F}, \mathbf{P})$ is called a **probability space**.

Throughout this book we denote by $\mathbf{E}$ or $\mathbf{E}^{\mathbf{P}}$ the expectation (integral) with respect to $\mathbf{P}$.

Given a family $\mathscr{C}$ of subsets of Ω, we denote the smallest σ-field including $\mathscr{C}$ by $\sigma(\mathscr{C})$:

$$\sigma(\mathscr{C}) = \bigcap_{\mathscr{G}} \mathscr{G},$$

where $\mathscr{G}$ runs over all σ-fields on Ω including $\mathscr{C}$.

If Ω is a topological space, $\mathscr{B}(\Omega)$ denotes the smallest σ-field containing all open subsets of Ω and is called the **Borel σ-field** on Ω.

Definition 1.1.3 Given a topological space E, a mapping $X : \Omega \to E$ is said to be $\mathscr{F}/\mathscr{B}(E)$-measurable if

$$X^{-1}(A) := \{\omega \in \Omega; X(\omega) \in A\} \in \mathscr{F}$$

holds for any $A \in \mathscr{B}(E)$. Such an X is called an E-valued **random variable**. The probability measure $\mathbf{P} \circ X^{-1}$ on E induced by X,

$$(\mathbf{P} \circ X^{-1})(A) = \mathbf{P}(X^{-1}(A)) \qquad (A \in \mathscr{B}(E)),$$

is called the **probability distribution** or **probability law** of X.

The purpose of this book is to study several kinds of analysis based on continuous stochastic processes. Here we introduce path spaces which play a fundamental role in such studies.

Definition 1.1.4 Let (E, d_E) be a complete separable metric space.
(1) For $T > 0$, $\mathbf{W}_T(E)$ stands for the set of E-valued continuous functions on $[0, T]$. $\mathbf{W}_T(E)$ is endowed with the topology of uniform convergence, or equivalently, with the distance function given by

$$d(w_1, w_2) = \max\{d_E(w_1(t), w_2(t)); 0 \leqq t \leqq T\},$$

which makes $\mathbf{W}_T(E)$ a complete separable metric space.
(2) The set of E-valued continuous functions on $[0, \infty)$ is denoted by $\mathbf{W}(E)$, and it is endowed with the topology of uniform convergence on compact sets, or equivalently, with the distance function given by

$$d(w_1, w_2) = \sum_{n=1}^{\infty} 2^{-n} \Big\{ \max_{0 \leqq t \leqq n} (d_E(w_1(t), w_2(t))) \wedge 1 \Big\},$$

by which $\mathbf{W}(E)$ is a complete separable metric space.

The Borel σ-fields with respect to the respective topologies are denoted by $\mathscr{B}(\mathbf{W}_T(E)), \mathscr{B}(\mathbf{W}(E))$.

Proposition 1.1.5 *Let $\mathscr{C}_T(E)$ be the totality of subsets of $\mathbf{W}_T(E)$ of the form*

$$\{w \in \mathbf{W}_T(E);\ w(t_1) \in A_1, w(t_2) \in A_2, \dots, w(t_n) \in A_n\} \tag{1.1.1}$$

with $0 < t_1 < t_2 < \cdots < t_n \leqq T$, $A_1, A_2, \dots, A_n \in \mathscr{B}(E)$. Then $\sigma(\mathscr{C}_T(E)) = \mathscr{B}(\mathbf{W}_T(E))$.

Proof For an open set G of E and for $t > 0$, $\{w \in \mathbf{W}_T(E); w(t) \in G\}$ is an open set of $\mathbf{W}_T(E)$. Hence $\sigma(\mathscr{C}_T(E)) \subset \mathscr{B}(\mathbf{W}_T(E))$.

To prove $\sigma(\mathscr{C}_T(E)) \supset \mathscr{B}(\mathbf{W}_T(E))$, it suffices to show

$$\Big\{w;\ \max_{0 \leqq t \leqq T} d_E(w(t), w_0(t)) \leqq \delta\Big\} \in \sigma(\mathscr{C}_T(E))$$

for any $w_0 \in \mathbf{W}_T(E)$ and $\delta > 0$. It is easily obtained from the following identity:

$$\Big\{w;\ \max_{0 \leqq t \leqq T} d_E(w(t), w_0(t)) \leqq \delta\Big\} = \bigcap_{0 \leqq r \leqq T,\, r \in \mathbb{Q}} \{w; d_E(w(r), w_0(r)) \leqq \delta\}. \qquad \square$$

We call a set of the form (1.1.1) a **Borel cylinder set**. We define the Borel cylinder sets of $\mathbf{W}(E)$ in the same way. Then, setting $\mathscr{C}(E) = \bigcup_{T>0} \mathscr{C}_T(E)$, we have $\sigma(\mathscr{C}(E)) = \mathscr{B}(\mathbf{W}(E))$.

Definition 1.1.6 Let $\mathbb{T}$ be $[0, T]$ $(T > 0)$ or $[0, \infty)$.
(1) A family $X = \{X(t)\}_{t \in \mathbb{T}}$ of E-valued random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ is called an E-valued **stochastic process**. When $\mathbb{T} = [0, \infty)$, we write $X = \{X(t)\}_{t \geqq 0}$.
(2) A stochastic process X is said to be continuous if for almost all $\omega \in \Omega$ an E-valued function $X(\omega) : \mathbb{T} \ni t \mapsto X(t)(\omega) \in E$ on $\mathbb{T}$ is continuous.
(3) For each $w \in \Omega$, the function $X(\omega)$ is called a **sample path**.

Definition 1.1.7 Let $\mathbb{T}$ be the same as above and $X = \{X(t)\}_{t \in \mathbb{T}}$ and $X' = \{X'(t)\}_{t \in \mathbb{T}}$ be stochastic processes defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. If $\mathbf{P}(X(t) = X'(t)) = 1$ for all $t \in \mathbb{T}$, X' is called a **modification** of X.

Throughout this book, $\langle x, y \rangle$ denotes the standard inner product of $x = (x^1, x^2, \dots, x^d)$, $y = (y^1, y^2, \dots, y^d) \in \mathbb{R}^d$ and $|x|$ denotes the norm of x:

$$\langle x, y \rangle = \sum_{i=1}^d x^i y^i, \qquad |x| = \Big\{\sum_{i=1}^d (x^i)^2\Big\}^{\frac{1}{2}}.$$

The next assertion is called **Kolmogorov's continuity theorem**.

Theorem 1.1.8 *Let $X = \{X(t)\}_{t \geqq 0}$ be an $\mathbb{R}^d$-valued stochastic process defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$, and assume that for any $T > 0$ there exist positive constants α, β, C such that*

$$\mathbf{E}[|X(t) - X(s)|^\alpha] \leqq C(t-s)^{1+\beta} \qquad (0 \leqq s < t \leqq T).$$

Then, there exists a modification X' of X which satisfies

$$P\Big(\lim_{h \downarrow 0} \sup_{\substack{0 \leqq s < t \leqq T \\ t-s<h}} \frac{|X'(t) - X'(s)|}{(t-s)^\varepsilon} = 0\Big) = 1 \tag{1.1.2}$$

for any $\varepsilon \in (0, \frac{\beta}{\alpha})$. In particular, X' is continuous.

For a proof, we refer to [56, 62]. Kolmogorov's continuity theorem for random fields (Theorem A.5.1) should also be studied.

1.2 Wiener Space

We fix $T > 0$ and let $W_T = W_T(\mathbb{R}^d)$ be the set of $\mathbb{R}^d$-valued continuous functions w on $[0, T]$ with $w(0) = 0$:

$$W_T = \{w : [0, T] \to \mathbb{R}^d;\ w \text{ is continuous and } w(0) = 0\}.$$

As was mentioned in the previous section, the space W_T is endowed with the topology of uniform convergence. We denote the Borel σ-field with respect to this topology by $\mathscr{B}(W_T)$, omitting $\mathbb{R}^d$.

The next theorem due to Wiener is one of the starting points of modern probability theory.

Theorem 1.2.1 *There exists a unique probability measure μ_T on W_T satisfying*

$$\mu_T(\{w \in W_T;\ w(t_1) \in A_1, w(t_2) \in A_2, \dots, w(t_n) \in A_n\})$$
$$= \int_{A_1} \mathrm{d}x_1 \int_{A_2} \mathrm{d}x_2 \cdots \int_{A_n} \mathrm{d}x_n \prod_{j=1}^n g(t_j - t_{j-1}, x_{j-1}, x_j) \tag{1.2.1}$$

for $0 = t_0 < t_1 < \cdots < t_n$ and $A_1, \dots, A_n \in \mathscr{B}(\mathbb{R}^d)$. Here $x_0 = 0$ and the function $g(t, x, y)$ is given by

$$g(t, x, y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|y-x|^2}{2t}}. \tag{1.2.2}$$

Definition 1.2.2 μ_T is called the d-dimensional **Wiener measure** and the probability space $(W_T, \mathscr{B}(W_T), \mu_T)$ is called the d-dimensional **Wiener space**

on $[0,T]$. Moreover, we call measurable functions (random variables) defined on the Wiener space **Wiener functionals**.

The function $g(t,x,y)$ is called the **Gauss kernel** or **heat kernel**. It is the fundamental solution for the heat equation. That is, it satisfies $\frac{\partial g}{\partial t} = \frac{1}{2}\Delta_x g$, Δ_x being the Laplacian acting on functions in x, and the function $u(t,x)$ defined by

$$u(t,x) = \int_{\mathbb{R}^d} g(t,x,y)f(y)\,\mathrm{d}y$$

for a bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_x u, \qquad \lim_{t\downarrow 0} u(t,x) = f(x).$$

Note that $u(t,x)$ is the expectation of the random variable $f(x+w(t))$ defined on $(W_T, \mathscr{B}(W_T), \mu_T)$. Moreover, $g(t,x,y)$ satisfies the equation

$$\int_{\mathbb{R}^d} g(t,x,y)g(s,y,z)\,\mathrm{d}y = g(t+s,x,z).$$

This relationship is known as the **Chapman–Kolmogorov equation**. We will discuss in detail the relationship between Brownian motions and Laplacians in Chapter 3.

When we need to write explicitly the dimension d of the Wiener space, we denote it by $(W_T^d, \mathscr{B}(W_T^d), \mu_T^d)$. We omit T when there is no fear of confusion.

The uniqueness of the Wiener measure follows from Proposition 1.1.5. We will mention a method to construct the Wiener measure after introducing a Hilbert space which plays an important role in stochastic analysis. But, before it, we show how to construct it via **random walks**.

It is sufficient to consider the case where $d = 1$ and $T = 1$. Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent and identically distributed random variables taking values ± 1 with probability $\frac{1}{2}$. The discrete time stochastic process $\{S_k\}_{k=0}^\infty$ defined by $S_0 = 0$ and $S_k = \xi_1 + \cdots + \xi_k$ is a simple random walk. We define a sequence $\{X_n(t)\}_{0\leqq t\leqq 1}$ $(n = 1,2,\ldots)$ of continuous stochastic processes by

$$X_n\Big(\frac{k}{n}\Big) = \frac{\xi_1 + \cdots + \xi_k}{\sqrt{n}}$$

for $t = \frac{k}{n}$ $(k = 1,2,\ldots,n)$ and, for $t \in [\frac{k}{n}, \frac{k+1}{n}]$,

$$X_n(t) = X_n\Big(\frac{k}{n}\Big) + n\Big(t - \frac{k}{n}\Big)\Big(X_n\Big(\frac{k+1}{n}\Big) - X_n\Big(\frac{k}{n}\Big)\Big).$$

Then the probability measure on W_1 induced by $\{X_n(t)\}$ converges weakly to the Wiener measure μ_1. This is the so-called Donsker's invariance principle (see, for example, [56]).

Let H_T be the subset of the d-dimensional Wiener space W_T which consists of $h = (h_1, \ldots, h_d) \in W_T$ such that each component h_i of h is absolutely continuous and has square-integrable derivative $\dot{h}_i$ on $[0, T]$. We call H_T the **Cameron–Martin subspace**. We denote it also by H_T^d.

The inner product $\langle h_1, h_2 \rangle_{H_T}$ is defined by

$$\langle h_1, h_2 \rangle_{H_T} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle \,\mathrm{d}t \qquad (h_1, h_2 \in H_T),$$

where $\langle \cdot\,, \,\cdot \rangle$ on the right hand side is the Euclidean inner product. Then H_T is a real separable Hilbert space. We denote the norm on H_T by

$$\|h\|_{H_T} = (\langle h, h \rangle_{H_T})^{\frac{1}{2}}.$$

H_T is embedded in W_T densely and continuously. Hence, by identifying the dual space H_T^* [1] with H_T by the Riesz theorem, we have

$$W_T^* \subset H_T^* = H_T \subset W_T \qquad \text{(densely and continuously).}$$

Proposition 1.2.3 *If $h \in H_T^*$ is of C^2-class, then $h \in W_T^*$ and, denoting by $\ddot{h}$ the second derivative, we have*

$$h(w) = \langle w(T), \dot{h}(T) \rangle - \int_0^T \langle w(t), \ddot{h}(t) \rangle \,\mathrm{d}t \qquad (w \in W_T). \tag{1.2.3}$$

Proof We define $\widetilde{h} \in W_T^*$ by the right hand side of (1.2.3). Then, for any $g \in H_T$,

$$\widetilde{h}(g) = \int_0^T \langle \dot{h}(t), \dot{g}(t) \rangle \,\mathrm{d}t = h(g),$$

which implies $\widetilde{h} = h$ in W_T^* because H_T is dense in W_T. □

Remark 1.2.4 If $h \in H_T^*$ is of piecewise C^2-class, that is, if there exists a sequence $0 = t_0 < t_1 < \cdots < t_N = T$ such that the restriction of h to (t_i, t_{i+1}) is of C^2-class, then $h \in W_T^*$ and

$$h(w) = \sum_{i=1}^N \Big\{ \langle w(t_i), \dot{h}(t_i - 0) \rangle - \langle w(t_{i-1}), \dot{h}(t_{i-1} + 0) \rangle - \int_{t_{i-1}}^{t_i} \langle w(t), \ddot{h}(t) \rangle \,\mathrm{d}t \Big\}.$$

Wiener constructed the Wiener measure by means of the Fourier expansion (see Example 1.2.6 below) and Lévy did so by the expansion using Schauder

[1] The set of continuous linear functionals on a linear topological space E is called the dual (or conjugate) space, which is denoted by E^* in this book.

functions (Example 1.2.7). We can understand these constructions in a unified manner by the celebrated Itô–Nisio theorem ([51]). We present it without proof.

Theorem 1.2.5 *Let* $\{h_n\}_{n=1}^\infty$ *be an orthonormal basis of the Cameron–Martin subspace* H_T *and* $\{X_n\}_{n=1}^\infty$ *be a sequence of independent and standard-normally distributed random variables defined on a probability space* $(\Omega, \mathscr{F}, \mathbf{P})$. *Then the sequence* $\{S_n\}_{n=1}^\infty$ *of* W_T*-valued random variables given by* $S_n = \sum_{i=1}^n X_i h_i$ *converges with probability* 1 *and the distribution of the limit (a probability measure on* W_T*) satisfies* (1.2.1).

Example 1.2.6 Let $d = 1$ and put

$$\psi_0(t) = \frac{t}{\sqrt{T}}, \quad \psi_n(t) = \frac{\sqrt{2T}}{n\pi}\sin\Big(\frac{n\pi t}{T}\Big) \quad (n \geqq 1).$$

Then $\{\psi_n\}_{n=0}^\infty$ is an orthonormal basis of H_T^1. Hence, letting $\{X_n\}_{n=0}^\infty$ be the same sequence as in Theorem 1.2.5 and setting

$$B(t) = \frac{X_0}{\sqrt{T}}t + \sum_{n=1}^\infty X_n \frac{\sqrt{2T}}{n\pi}\sin\Big(\frac{n\pi t}{T}\Big),$$

we see that the probability measure on W_T^1 induced by $\{B(t)\}_{0\leqq t\leqq T}$ is the Wiener measure. This is called the Fourier expansion of Brownian motion.

Example 1.2.7 Letting $\boldsymbol{e}^1, \dots, \boldsymbol{e}^d$ be the standard basis of $\mathbb{R}^d$, we define the $\mathbb{R}^d$-valued functions $\{h_j^i\}_{i=1,\dots,d,\, j=0,1,\dots}$ by $h_j^i(0) = 0$,

$$\dot{h}_0^i(t) = \frac{1}{\sqrt{T}}\boldsymbol{e}^i,$$

for $j = 0$ and

$$\dot{h}_{2^n+k}^i(t) = \begin{cases} \sqrt{\frac{2^n}{T}}\boldsymbol{e}^i & (\frac{k}{2^n}T \leqq t \leqq \frac{2k+1}{2^{n+1}}T) \\ -\sqrt{\frac{2^n}{T}}\boldsymbol{e}^i & (\frac{2k+1}{2^{n+1}}T \leqq t \leqq \frac{k+1}{2^n}T) \\ 0 & (\text{otherwise}) \end{cases}$$

for $j = 2^n + k$ $(k = 0, 1, \dots, 2^n - 1,\ n = 0, 1, \dots)$. $\{h_j^i\}$ forms an orthonormal basis of H_T^d. Lévy constructed the Wiener measure by using $\{h_j^i\}$.[2]

[2] $\dot{h}_j^i$ is called the **Haar function** and h_j^i is called the **Schauder function**.

The function $t \mapsto h^i_{2^n+k}(t)$ vanishes outside the interval $[\frac{k}{2^n}T, \frac{k+1}{2^n}T]$. It is piecewise linear on this interval and takes the maximum $\sqrt{\frac{T}{2^{n+2}}}$ at the midpoint. Moreover, since it is of piecewise C^2-class, it belongs to W_T^*.

We define $\ell_m(w) \in W_T$ by

$$\ell_m(w)(t) = \sum_{i=1}^{d} \sum_{j=0}^{2^m-1} \langle h^i_j, w\rangle h^i_j(t).$$

Then the function $t \mapsto \ell_m(w)(t)$ is piecewise linear and $\ell_m(w)(\frac{k}{2^m}T) = w(\frac{k}{2^m}T)$. Hence $\ell_m(w)$ is the piecewise linear function which connects $w(\frac{k}{2^m}T)$ $(k = 0, 1, \dots, 2^m)$ in order and Lévy's construction gives us the piecewise linear approximation of $w \in W$.

Next we consider the path space with the time interval $[0, \infty)$:

$$W = \{w : [0, \infty) \to \mathbb{R}^d;\ w \text{ is continuous and } w(0) = 0\}.$$

As mentioned in the previous section, we endow W with the topology of uniform convergence on compact sets. Note that a similar assertion to Proposition 1.1.5 holds. In this case we call the following space H the **Cameron–Martin subspace**:

$$H = \{h \in W;\ h \text{ is absolutely continuous and } \dot{h} \in L^2([0, \infty))\}.$$

For $T > 0$ we let $\varphi_T(w)$ be the restriction of $w \in W$ to $[0, T]$. Then, there exists a probability measure μ on W whose image measure under φ_T is the Wiener measure on W_T. We also call μ the Wiener measure and the probability space $(W, \mathscr{B}(W), \mu)$ the d-dimensional Wiener space on $[0, \infty)$.

We show some basic properties of the Wiener measure.

Theorem 1.2.8 (1) *Define the transforms* ψ_c $(c > 0)$, ϕ_s $(s > 0)$ *and* φ_Q (Q *is a d-dimensional orthogonal matrix*) *on* W *by*

$$\psi_c(w)(t) = \frac{1}{c}w(c^2 t), \quad \phi_s(w)(t) = w(s + t) - w(s), \quad \varphi_Q(w)(t) = Qw(t).$$

Then μ *is invariant under* $\psi_c, \phi_s, \varphi_Q$.
(2) *Define the transform* Φ_T *on* W_T *by*

$$\Phi_T(w)(t) = w(T - t) - w(T) \quad (0 \leqq t \leqq T).$$

Then μ_T *is invariant under* Φ_T.

Proof To see the invariance under ψ_c, we have only to show

$$\mu\left(\left\{w;\ \frac{1}{c}w(c^2t_1) \in A_1, \frac{1}{c}w(c^2t_2) \in A_2, \ldots, \frac{1}{c}w(c^2t_n) \in A_n\right\}\right)$$
$$= \mu(\{w;\ w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\})$$

for $t_1 < t_2 < \cdots < t_n$, $A_1, A_2, \ldots, A_n \in \mathscr{B}(\mathbb{R}^d)$ by using the identity $g(t,x,y) = c^d \cdot g(c^2t, cx, cy)$ $(c > 0)$. The other assertions are shown similarly. □

Remark 1.2.9 Set $\Psi(w)(0) = 0$, $\Psi(w)(t) = tw(\frac{1}{t})$ $(t > 0)$. Then $\Psi(w)$ is continuous at $t = 0$ almost surely under the Wiener measure. Hence, we may regard Ψ as a transform of the Wiener space and can prove that the Wiener measure is invariant under Ψ in the same way as in Theorem 1.2.8.

Definition 1.2.10 A d-dimensional continuous stochastic process $\{X(t)\}_{t\geqq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ is called a d-dimensional **Brownian motion** (or **Wiener process**) starting from 0 if $X(0) = 0$ and its probability distribution on W is the Wiener measure.

1.3 Filtered Probability Space, Adapted Stochastic Process

Let $(W, \mathscr{B}(W), \mu)$ be the d-dimensional Wiener space on $[0,\infty)$ and define a function $\theta(s)$ $(s \geqq 0)$ on W by $\theta(s)(w) = w(s)$. Then the smallest σ-field which makes the behavior of each $w \in W$ up to time t measurable is

$$\mathscr{B}_t^0 := \sigma(\{\theta(s)^{-1}(A); A \in \mathscr{B}(\mathbb{R}^d),\ 0 \leqq s \leqq t\}) \tag{1.3.1}$$

and it forms an increasing sequence $\{\mathscr{B}_t^0\}_{t\geqq 0}$ of sub-σ-fields of $\mathscr{B}(W)$. Moreover, setting

$$\mathscr{B}_t = \bigcap_{u>t} \mathscr{B}_u^0,$$

we see that $\{\mathscr{B}_t\}_{t\geqq 0}$ is right-continuous, that is, $\mathscr{B}_{t+0} := \bigcap_{u>t} \mathscr{B}_u = \mathscr{B}_t$.[3]

The stochastic process $\{\theta(t)\}_{t\geqq 0}$ defined on the Wiener space W is called the **coordinate process**.

Definition 1.3.1 A quartet $(\Omega, \mathscr{F}, \mathbf{P}, \{\mathscr{F}_t\})$ of a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and an increasing sequence $\{\mathscr{F}_t\}_{t\geqq 0}$ of sub-σ-fields of $\mathscr{F}$ is called a **filtered probability space**.

[3] $\{\mathscr{B}_t^0\}$ is not right-continuous. For example, for a function ϕ on $[0,\infty)$, the random variable $\limsup_{u\downarrow t} \frac{w(u)-w(t)}{\phi(u-t)}$ is not $\mathscr{B}_t^0$-measurable, but it is $\mathscr{B}_t$-measurable.

$\{\mathscr{F}_t\}$ is called a **filtration**. If $\{\mathscr{F}_t\}$ is right-continuous and each $\mathscr{F}_t$ contains all **P**-null sets, $\{\mathscr{F}_t\}$ is said to satisfy the **usual condition**.

Next we mention the measurability of stochastic processes. While the purpose of this book is to develop analysis of continuous stochastic processes, we need to consider stochastic processes with discontinuous paths. Intuitive understanding is enough for our purpose and we do not discuss in detail but refer to [45, 56, 86, 114] and so on.

Let E be a complete separable metric space.

Definition 1.3.2 Let $X = \{X(t)\}_{t \geqq 0}$ be an E-valued stochastic process defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{P}, \{\mathscr{F}_t\})$.
(1) X is called **measurable** if $X(t)(\omega)$ is $\mathscr{B}([0,\infty)) \times \mathscr{F}$-measurable as a function of (t,ω).
(2) X is $\{\mathscr{F}_t\}$-**adapted** if the E-valued random variable $X(t)$ is $\mathscr{F}_t$-measurable for each t.
(3) X is $\{\mathscr{F}_t\}$-**progressively measurable** if the map $[0,t] \times \Omega \ni (s,\omega) \mapsto X(s)(\omega) \in E$ is $\mathscr{B}([0,t]) \times \mathscr{F}_t$-measurable for each t.

We often write $X(s,\omega)$ for the value $X(s)(\omega)$ of the sample process at s, regarding it as a function in two variables (s,ω).

Proposition 1.3.3 *If an E-valued $\{\mathscr{F}_t\}$-adapted stochastic process X defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{P}, \{\mathscr{F}_t\})$ has right-continuous paths, that is, if for any $\omega \in \Omega$ the map $t \mapsto X(t)(\omega)$ is right-continuous, then X is $\{\mathscr{F}_t\}$-progressively measurable.*

Proof Fix $t > 0$ and put for $n = 1, 2, \ldots$

$$X^{(n)}(0) = X(0),$$
$$X^{(n)}(s) = \sum_{j=0}^{\infty} X\Big(\frac{(j+1)t}{n}\Big) \mathbf{1}_{[\frac{jt}{n}, \frac{(j+1)t}{n})}(s) \qquad (s > 0).$$

Then, letting $j = j(s)$ $(s \in [0,t])$ be the integer such that $\frac{jt}{n} < s \leqq \frac{(j+1)t}{n}$, we see that both

$$[0,t] \times \Omega \ni (s,\omega) \mapsto X\Big(\frac{(j+1)t}{n}\Big)(\omega) \quad \text{and} \quad \mathbf{1}_{[\frac{jt}{n}, \frac{(j+1)t}{n})}(s)$$

are $\mathscr{B}([0,t]) \times \mathscr{F}_t$-measurable. Since $\lim_{n\to\infty} X^{(n)}(s)(\omega) = X(s)(\omega)$ for any (s,ω) by the right-continuity of $\{X(s)\}$, $X(s)(\omega)$ is $\mathscr{B}([0,t]) \times \mathscr{F}_t$-measurable. □

We end this section with an explanation on stopping times.