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1

Representation formulae for singular solutions of polyharmonic and parabolic inequalities

1.1 Introduction

This chapter is devoted to various integral representation formulae for singular solutions of polyharmonic and parabolic inequalities. This topic forms the building block in our study of isolated singularities for differential inequalities. Although integral representation formulae have been around for two centuries (a remarkable example is the famous Poisson integral formula for the Laplace's equation in the ball), in nonlinear PDEs they have been employed only in the last few decades. In this chapter, the reader will be gradually introduced to the topic of integral representation of distributional or classical solutions of linear differential inequalities. We start with a basic representation formula for superharmonic functions in a punctured ball. Then we extend such a result to solutions of $-\Delta^m u \geq 0$ in $B_1(0) \setminus \{0\} \subset \mathbb{R}^n$, $m, n \geq 1$. Finally, the integral representation for singular solutions corresponding to the heat operator is discussed. A common feature of all these results is that the integral operator in the representation formulae of the solution contains as a kernel the fundamental solution of the differential operator under consideration.

1.2 Harmonic inequalities in the punctured ball

In this section we consider C^2 nonnegative solutions of the harmonic inequality

$$-\Delta u \geq 0 \quad \text{in} \quad B_2(0) \setminus \{0\} \subset \mathbb{R}^n, \ n \geq 2. \tag{1.2.1}$$

According to the following theorem, these solutions satisfy representation formula (1.2.2) below.

Theorem 1.1 *Suppose u is a nonnegative C^2 solution of* (1.2.1) *and let $f = -\Delta u$. Then, $u, f \in L^1(B_1(0))$ and there exist a nonnegative constant m and a harmonic function $h : B_1(0) \to \mathbb{R}$ such that*

$$u(x) = m\Phi(|x|) + N(x) + h(x) \quad \text{in} \quad B_1(0) \setminus \{0\}, \tag{1.2.2}$$

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where

$$N(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n} \displaystyle\int_{|y|<1} |x-y|^{2-n} f(y)dy & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \displaystyle\int_{|y|<1} \log\left(\dfrac{2}{|x-y|}\right) f(y)dy & \text{if } n = 2, \end{cases} \tag{1.2.3}$$

and

$$\Phi(r) = \begin{cases} r^{2-n} & \text{if } n \geq 3, \\ \log \dfrac{1}{r} & \text{if } n = 2. \end{cases} \tag{1.2.4}$$

In (1.2.3), ω_n is the volume of the unit ball in $\mathbb{R}^n$.

For the proof of Theorem 1.1 we will need the following lemma.

Lemma 1.2 *Suppose v is a harmonic function in $B_1(0) \setminus \{0\}$ such that*

$$\int_{|x|<\varepsilon} |v(x)|dx = o(\varepsilon) \quad \text{as } \varepsilon \to 0^+. \tag{1.2.5}$$

Then $v - \beta\Phi(|x|)$ has a harmonic extension to $B_1(0)$ for some $\beta \in \mathbb{R}$.

Proof Let $\bar{v}(r)$ be the spherical average of v over the sphere $\partial B_r(0)$. Since v is harmonic in $B_1(0) \setminus \{0\}$, there exist $b, \beta \in \mathbb{R}$ such that

$$\bar{v}(r) = \beta\Phi(r) + b \quad \text{for all } 0 < r < 1. \tag{1.2.6}$$

Let $\eta \in C^\infty(\mathbb{R})$ be a decreasing function such that $\eta(t) = 1$ for $t \leq 1/2$ and $\eta(t) = 0$ for $t > 1$. For $\varepsilon > 0$ define $\psi_\varepsilon : \mathbb{R}^n \to [0, 1]$ by $\psi_\varepsilon(x) = \eta(|x|/\varepsilon)$.

Let now $\varphi \in C_0^\infty(B_1(0))$ and $\widehat{\varphi} = \varphi - \varphi(0)$. Then for small $\varepsilon > 0$ we have

$$\begin{aligned} \int (v - \bar{v}(|x|))\Delta\varphi &= \int (v - \bar{v}(|x|))[\Delta(\varphi\psi_\varepsilon) + \Delta(\varphi(1-\psi_\varepsilon))] \\ &= \int (v - \bar{v}(|x|))\Delta(\varphi\psi_\varepsilon) = I_1(\varepsilon) + \varphi(0)I_2(\varepsilon), \end{aligned} \tag{1.2.7}$$

where

$$I_1(\varepsilon) = \int (v - \bar{v})\Delta(\widehat{\varphi}\psi_\varepsilon) \quad \text{and} \quad I_2(\varepsilon) = \int (v - \bar{v})\Delta\psi_\varepsilon.$$

Since $\Delta\psi_\varepsilon$ is radial about the origin, $I_2(\varepsilon) = 0$ for small $\varepsilon > 0$. Also, since $\max_{|x|<\varepsilon} |\Delta(\widehat{\varphi}\psi_\varepsilon)| = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$, it follows from (1.2.5) and (1.2.6) that $I_1(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Hence letting $\varepsilon \to 0^+$ in (1.2.7) completes the proof. □

Proof of Theorem 1.1 Let $\bar{u}(r)$ denote the average of u over $\partial B_r(0)$. Averaging and integrating (1.2.1) we get

$$r^{n-1}\bar{u}'(r) = \bar{u}'(1) + \frac{1}{n\omega_n} \int_{r<|x|<1} f(x)dx \quad \text{for all } 0 < r < 1. \tag{1.2.8}$$

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We claim $f \in L^1(B_1(0))$ for otherwise there exists $r_0 \in (0, 1)$ such that the right-hand side in (1.2.8) is larger than 1 for $0 < r < r_0$ and hence

$$\bar{u}(r_0) - \bar{u}(r) \geq \begin{cases} \frac{1}{n-2}\left(r^{2-n} - r_0^{2-n}\right) & \text{if } n \geq 3, \\ \log \frac{r_0}{r} & \text{if } n = 2, \end{cases}$$

for $0 < r < r_0$. In particular, the above inequality implies $\bar{u}(r_0) - \bar{u}(r) \to \infty$ as $r \to 0^+$ which contradicts $u \geq 0$ in $B_1(0) \setminus \{0\}$. Hence $f \in L^1(B_1(0))$ and thus from (1.2.8) there exists a finite constant m such that

$$\frac{\bar{u}(r)}{\Phi(r)} = m + o(1) \quad \text{as } r \to 0^+. \tag{1.2.9}$$

Since $u \geq 0$, from (1.2.9) we derive $m \geq 0$ and as $\varepsilon \to 0^+$ we have

$$\int_{|x|<\varepsilon} u(x)dx = \begin{cases} O(\varepsilon^2) & \text{if } n \geq 3, \\ O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right) & \text{if } n = 2. \end{cases} \tag{1.2.10}$$

In particular, (1.2.10) implies $u \in L^1(B_1(0))$. Let next $N(x)$ be defined by (1.2.3) for all $x \in \mathbb{R}^n \setminus \{0\}$. Then $N \in C^1(\mathbb{R}^n \setminus \{0\}) \cap L^1(B_1(0))$ and

$$-\Delta N = f \quad \text{in } \mathcal{D}'(B_1(0)).$$

Using the identity

$$\frac{1}{|\partial B_r(0)|} \int_{|x|=r} \Phi(|x-y|)dS = \begin{cases} \Phi(|y|) & \text{if } |y| > r, \\ \Phi(r) & \text{if } |y| < r, \end{cases}$$

we easily find

$$\bar{N}(r) = o(\Phi(r)) \quad \text{as } r \to 0^+. \tag{1.2.11}$$

By (1.2.10) and (1.2.11) we now obtain

$$\int_{|x|<\varepsilon} |u(x) - N(x)|dx \leq \int_{|x|<\varepsilon} u(x)dx + \int_{|x|<\varepsilon} N(x)dx = \begin{cases} O(\varepsilon^2) & \text{if } n \geq 3, \\ O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right) & \text{if } n = 2, \end{cases}$$

as $\varepsilon \to 0^+$. It therefore follows from Lemma 1.2 that $u - N(x) - \beta\Phi(|x|)$ has a harmonic extension to $B_1(0)$. That is,

$$u = \beta\Phi(|x|) + N(x) + h(x) \quad \text{in } B_1(0) \setminus \{0\}, \tag{1.2.12}$$

where $h : B_1(0) \to \mathbb{R}$ is a harmonic function. Averaging (1.2.12) and using (1.2.11) and (1.2.9) we get $\beta = m \geq 0$ which completes our proof. □

The L^1-regularity of Δu is still true when dealing with distributional solutions that exhibit higher dimensional singularity set. This is illustrated by the following result.

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Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Sigma \subset\subset \Omega$ be a closed set of zero Newtonian capacity and assume that $u, f \in L^1_{loc}(\Omega \setminus \Sigma)$ are two nonnegative functions such that*

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

Then $u, f \in L^1_{loc}(\Omega)$ and

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

Proof For $j \geq 1$ let $u_j = \min\{u, j\}$ and $f_j = f\chi_{\{u<j\}}$. By Kato's inequality [69] we have

$$-\Delta u_j \geq f_j \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma). \tag{1.2.13}$$

Since $-\Delta u_j$ is a nonnegative distribution in $\Omega \setminus \Sigma$, it can be extended to a nonnegative measure on $\Omega \setminus \Sigma$. Also, the boundedness of u_j combined with a Gagliardo–Nirenberg type inequality yields $u_j \in H^1_{loc}(\Omega \setminus \Sigma)$. We claim that $u_j \in H^1_{loc}(\Omega)$. To see this, let $\phi \in C^\infty_c(\Omega)$ and let $\{\phi_k\} \subset C^\infty_c(\Omega \setminus \Sigma)$ be such that $\phi_k \to \phi$ in $H^1(\Omega)$. This is possible since $\text{cap}_\Omega(\Sigma) = 0$ (e.g., $\phi_k = \phi(1 - \chi_k)$ where $\chi_k = 1$ near Σ and $\|\chi_k\|_{H^1} \to 0$). We then have

$$\begin{aligned}
\int |\nabla u_j|^2 \phi_k^2 &\leq -e^j \int \phi_k^2 \nabla(e^{-u_j}) \cdot \nabla u_j \\
&= e^j \left(2 \int e^{-u_j} \phi_k \nabla\phi_k \cdot \nabla u_j + \int e^{-u_j} \Delta u_j \phi_k^2 \right) \\
&\leq 2e^{2j} \int e^{-u_j} |\nabla \phi_k|^2 + \frac{1}{2} \int e^{-u_j} |\nabla u_j|^2 \phi_k^2 \\
&\leq 2e^{2j} \int e^{-u_j} |\nabla \phi_k|^2 + \frac{1}{2} \int |\nabla u_j|^2 \phi_k^2
\end{aligned}$$

Hence

$$\int |\nabla(u_j \phi_k)|^2 \leq C_j \int |\nabla \phi_k|^2 \quad \text{for all } j, k \geq 1,$$

where $C_j > 0$. Passing to the limit with $k \to \infty$ in the above inequality we find $u_j \in H^1_{loc}(\Omega)$ for all $j \geq 1$. We next claim that

$$-\Delta u_j \geq f_j \quad \text{in } \mathcal{D}'(\Omega). \tag{1.2.14}$$

Indeed, let us notice first that from (1.2.13) we have

$$\int u_j(-\Delta \phi_k) \geq \int f_j \phi_k. \tag{1.2.15}$$

Using $u_j \in H^1_{loc}(\Omega)$ we deduce

$$\int u_j(-\Delta\phi_k) = \int \nabla u_j \cdot \nabla \phi_k \to \int \nabla u_j \cdot \nabla \phi = -\int u_j \Delta \phi \quad \text{as } k \to \infty.$$

Passing to the limit with $k \to \infty$ in (1.2.15) we obtain (1.2.14).

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From the fact that u_j is superharmonic in Ω, it follows that for almost all $x \in \Omega$ and for any ball $B \subset\subset \Omega$ centered at x we have

$$u_j(x) \geq \frac{1}{|B|} \int_B u_j(y) dy. \tag{1.2.16}$$

Since $u \in L^1_{loc}(\Omega \setminus \Sigma)$ and $|\Sigma| = 0$, we deduce that u is finite a.e. in Ω and $u_j \to u$ as $j \to \infty$. By Fatou's lemma and (1.2.16) we deduce that for almost every ball $B \subset\subset \Omega$ we have $u \in L^1(B)$, which means that $u \in L^1_{loc}(\Omega)$. Thus, we can pass to the limit in (1.2.14) and conclude that $f \in L^1_{loc}(\Omega)$ and

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega). \qquad \square$$

1.3 Polyharmonic inequalities in the punctured ball

This section is devoted to weak nonnegative solutions u of the polyharmonic inequality

$$-\Delta^m u \geq 0 \quad \text{in } B_{2R}(0) \setminus \{0\} \subset \mathbb{R}^n, \tag{1.3.1}$$

where $n \geq 2$ and $m \geq 1$ are integers.

A fundamental solution of the polyharmonic operator Δ^m in $\mathbb{R}^n$, where $n \geq 2$ and $m \geq 1$ are integers, is given by

$$\Phi(x) := a \begin{cases} (-1)^m |x|^{2m-n} & \text{if } 2 \leq 2m < n, & (1.3.2) \\ (-1)^{\frac{n-1}{2}} |x|^{2m-n} & \text{if } 3 \leq n < 2m \text{ and } n \text{ is odd}, & (1.3.3) \\ (-1)^{\frac{n}{2}} |x|^{2m-n} \log \frac{5}{|x|} & \text{if } 2 \leq n \leq 2m \text{ and } n \text{ is even}, & (1.3.4) \end{cases}$$

where $a = a(m, n)$ is a *positive* constant. In the sense of distributions, $\Delta^m \Phi = \delta$, where δ is the Dirac mass at the origin in $\mathbb{R}^n$. For $x \neq 0$ and $y \neq x$, let

$$\Psi(x, y) = \Phi(x - y) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x) \tag{1.3.5}$$

be the error in approximating $\Phi(x - y)$ with the partial sum of degree $2m - 3$ of the Taylor series of Φ at x.

The following theorem gives representation formula (1.3.8) for nonnegative solutions u of inequality (1.3.1).

Theorem 1.4 *Suppose $u, f \in L^1_{loc}(B_{2R}(0) \setminus \{0\} \subset \mathbb{R}^n)$ are nonnegative functions and*

$$-\Delta^m u = f \quad \text{in } \mathcal{D}'(B_{2R}(0) \setminus \{0\}), \tag{1.3.6}$$

where $n \geq 2$ and $m \geq 1$ are integers and $R \in (0, 1]$. Then $u \in L^1(B_1(0))$,

$$\int_{|x|<R} |x|^{2m-2} f(x)\, dx < \infty \tag{1.3.7}$$

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and

$$u = N + h + \sum_{|\alpha| \le 2m-2} a_\alpha D^\alpha \Phi \quad in \quad L^1(B_R(0)) \tag{1.3.8}$$

where a_α are constants, $h \in C^\infty(B_R(0))$ is a solution of

$$\Delta^m h = 0 \quad in \quad B_R(0),$$

and

$$N(x) = \int_{|y|<R} \Psi(x, y)\Delta^m u(y)\, dy \quad for \quad x \neq 0. \tag{1.3.9}$$

When $m = 1$, (1.3.8) becomes

$$u = N + h + a_0 \Phi_1 \quad \text{in} \quad L^1(B_R(0)),$$

where

$$N(x) = \int_{|y|<R} \Phi_1(x - y)\Delta u(y)\, dy$$

and Φ_1 is the fundamental solution of the Laplacian in $\mathbb{R}^n$. Thus, when $m = 1$, Theorem 1.4 reduces to Theorem 1.1.

1.3.1 Preliminary results

In this section we provide some results which will be needed for the proof of Theorem 1.4. Propositions 1.11, 1.13, and 1.15 below are of independent interest. We assume throughout this section that $n \ge 2$ and $m \ge 1$ are integers and

$$A = \{x \in \mathbb{R}^n : a < |x| < b\} \quad \text{where } 0 \le a < b.$$

Definition 1.5 If $u \in L^1_{\text{loc}}(A)$ then by Fubini's theorem

$$\bar{u}(r) := \frac{1}{n\omega_n r^{n-1}} \int_{|x|=r} u(x)\, dS_x$$

is defined and finite for almost all $r \in (a, b)$ and thus we can define $\hat{u}(x)$ for almost all $x \in A$ by

$$\hat{u}(x) := \bar{u}(|x|).$$

Definition 1.6 We say $u \in L^1_{\text{loc}}(A)$ is radial if $u = \hat{u}$ almost everywhere in A.

Lemma 1.7 *Suppose $u \in L^1_{loc}(A)$. Then $\hat{u} \in L^1_{loc}(A)$ and $\hat{u}$ is radial.*

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Proof For $a < \alpha < \beta < b$ we have

$$\int\limits_{\alpha<|x|<\beta} |\hat{u}(x)|\,dx = \int\limits_{\alpha}^{\beta} \int\limits_{|x|=r} |\bar{u}(|x|)|\,dS_x\,dr = \int\limits_{\alpha}^{\beta} n\omega_n r^{n-1}|\bar{u}(r)|\,dr$$
$$\le \int\limits_{\alpha}^{\beta} \int\limits_{|x|=r} |u(x)|\,dS_x\,dr = \int\limits_{\alpha<|x|<\beta} |u(x)|\,dx.$$

Thus $\hat{u} \in L^1_{\rm loc}(A)$.

For almost all $r \in (a, b)$ we have

$$\bar{\hat{u}}(r) = \frac{1}{n\omega_n r^{n-1}} \int\limits_{|x|=r} \hat{u}(x)\,dS_x = \frac{1}{n\omega_n r^{n-1}} \int\limits_{|x|=r} \bar{u}(|x|)\,dS_x = \bar{u}(r).$$

Thus $\hat{\hat{u}}(x) = \bar{\hat{u}}(|x|) = \bar{u}(|x|) = \hat{u}(x)$ for almost all $x \in A$. □

Lemma 1.8 *Let $u \in C^{2m}(A)$, then*

$$\Delta^m \hat{u} = \widehat{\Delta^m u} \quad \text{in } A. \tag{1.3.10}$$

Proof Since

$$\bar{u}(r) = \frac{1}{n\omega_n} \int\limits_{|\theta|=1} u(r\theta)\,dS_\theta \quad \text{for } a < r < b,$$

we see that $\bar{u} : (a, b) \to \mathbb{R}$ and $\hat{u} : A \to \mathbb{R}$ are C^{2m} functions.

We use induction to prove (1.3.10). Suppose first that $m = 1$. Then for all $\varphi \in C_0^\infty(a, b)$ we have

$$n\omega_n \int\limits_a^b r^{n-1}\overline{\Delta u}(r)\varphi(r)\,dr = \int\limits_a^b \left(\int\limits_{|x|=r} \Delta u(x)\,dS_x \right) \varphi(r)\,dr$$
$$= \int\limits_{a<|x|<b} \Delta u(x)\varphi(|x|)\,dx = \int\limits_{a<|x|<b} u(x)\Delta_x(\varphi(|x|))\,dx$$
$$= \int\limits_a^b \left(\int\limits_{|x|=r} u(x)\,dS_x \right) \left(\varphi''(r) + \tfrac{n-1}{r}\varphi'(r)\right)\,dr$$
$$= n\omega_n \int\limits_a^b \bar{u}(r)(r^{n-1}\varphi'(r))'\,dr = -n\omega_n \int\limits_a^b r^{n-1}\bar{u}'(r)\varphi'(r)\,dr$$
$$= n\omega_n \int\limits_a^b (r^{n-1}\bar{u}'(r))'\varphi(r)\,dr.$$

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Thus for $|x| = r \in (a, b)$ we have

$$\widehat{\Delta u}(x) = \overline{\Delta u}(r) = \frac{1}{r^{n-1}}(r^{n-1}\bar{u}'(r)) = \bar{u}''(r) + \frac{n-1}{r}\bar{u}'(r)$$
$$= \Delta_x(\bar{u}(|x|)) = \Delta\hat{u}(x).$$

Suppose (1.3.10) holds for $m - 1$. Then by the $m = 1$ case we have

$$\Delta^m\hat{u} = \Delta^{m-1}(\Delta\hat{u}) = \Delta^{m-1}\widehat{\Delta u} = \widehat{\Delta^m u}. \qquad \square$$

Lemma 1.9 *Let $u \in L^1_{loc}(A)$. Then*

$$\int \hat{u}\varphi = \int u\hat{\varphi} \quad \textit{for all } \varphi \in C_0^\infty(A).$$

Proof For all $\varphi \in C_0^\infty(A)$ we have

$$\int \hat{u}\varphi\, dx = \int_a^b \int_{|x|=r} \hat{u}(x)\varphi(x)\, dS_x\, dr = \int_a^b \bar{u}(r) n\omega_n r^{n-1}\bar{\varphi}(r)\, dr$$

and

$$\int u\hat{\varphi}\, dx = \int_a^b \int_{|x|=r} u(x)\hat{\varphi}(x)\, dS_x\, dr = \int_a^b \bar{\varphi}(r) n\omega_n r^{n-1}\bar{u}(r)\, dr. \qquad \square$$

Lemma 1.10 *Suppose $u, \Delta^m u \in L^1_{loc}(A)$ and u is radial. Then $\Delta^m u$ is radial.*

Proof For all $\varphi \in C_0^\infty(A)$ it follows from Lemmas 1.9 and 1.8 that

$$\int (\Delta^m u)\varphi = \int u\Delta^m\varphi = \int \hat{u}\Delta^m\varphi = \int u\widehat{\Delta^m\varphi} = \int u\Delta^m\hat{\varphi}$$
$$= \int (\Delta^m u)\hat{\varphi} = \int (\widehat{\Delta^m u})\varphi.$$

Thus $\Delta^m u = \widehat{\Delta^m u}$ almost everywhere in A. $\square$

Proposition 1.11 *Suppose $u, \Delta^m u \in L^1_{loc}(A)$. Then*

$$\Delta^m\hat{u} = \widehat{\Delta^m u} \quad \textit{in } \mathcal{D}'(A). \tag{1.3.11}$$

Proof For all $\varphi \in C_0^\infty(A)$ it follows from Lemmas 1.9 and 1.8 that

$$\int \widehat{\Delta^m u}\varphi = \int (\Delta^m u)\hat{\varphi} = \int u(\Delta^m\hat{\varphi}) = \int u\widehat{\Delta^m\varphi} = \int \hat{u}(\Delta^m\varphi)$$

which implies (1.3.11). $\square$

Corollary 1.12 *Suppose $u, \Delta^m u \in L^1_{loc}(A)$. Then $\overline{\Delta^m u}(r) = (\Delta^m\bar{u})(r)$ for almost all $r \in (a, b)$ where*

$$(\Delta^m\bar{u})(r) := (\Delta^m\hat{u}(x))|_{|x|=r} = \Delta_x^m\bar{u}(|x|)|_{|x|=r}.$$

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Proof For almost all $r \in (a, b)$, if $|x| = r$ then by Proposition 1.11 and Definition 1.5 we have

$$\begin{aligned}\overline{\Delta^m u}(r) &= \overline{\Delta^m u}(|x|) = \widehat{\Delta^m u}(x)\\ &= (\Delta^m \hat{u})(x) = (\Delta^m \bar{u})(r).\end{aligned}$$

□

Proposition 1.13 *Suppose $u, \Delta^m u \in L^1_{loc}(A)$. Then $\bar{u}$ is $2m - 1$ times continuously differentiable on the interval (a, b) and $\bar{u}^{(2m-1)}$ is absolutely continuous on every closed subinterval of (a, b). Moreover,*

$$\Delta^m_x \bar{u}(|x|)|_{|x|=r} = \overline{\Delta^m u}(r) \quad \text{for almost all } r \in (a, b).$$

Proof Let $[\alpha, \beta]$ be a closed subinterval of (a, b). By Lemma 1.7 and Definition 1.5,

$$\begin{aligned}\infty > \int\limits_{\alpha<|x|<\beta} |\widehat{\Delta^m u}(x)|\, dx &= \int\limits_{\alpha}^{\beta} \int\limits_{|x|=\rho} |\overline{\Delta^m u}(|x|)|\, dS_x\, d\rho\\ &= \int\limits_{\alpha}^{\beta} n\omega_n \rho^{n-1} |\overline{\Delta^m u}(\rho)|\, d\rho.\end{aligned}$$

Hence $\int_r^\beta \rho^{n-1}\overline{\Delta^m u}(\rho)\, d\rho$ is absolutely continuous on $[\alpha, \beta]$. Thus defining Nf for $f \in L^1_{\text{loc}}(a, b)$ by

$$(Nf)(r) = \int\limits_{r}^{\beta} \frac{1}{s^{n-1}} \int\limits_{s}^{\beta} \rho^{n-1} f(\rho)\, d\rho\, ds$$

we have the function $v : (a, b) \to \mathbb{R}$ defined by

$$v(r) = (N^m \overline{\Delta^m u})(r)$$

is $2m - 1$ times continuously differentiable on (a, b), $v^{(2m-1)}$ is absolutely continuous on $[\alpha, \beta]$ and for almost all $r \in [\alpha, \beta]$ if $|x| = r$ then by Proposition 1.11 and Definition 1.5 we obtain

$$\begin{aligned}\Delta^m_x v(|x|) &= \overline{\Delta^m u}(|x|)\\ &= \widehat{\Delta^m u}(x) = (\Delta^m \hat{u})(x).\end{aligned} \tag{1.3.12}$$

Hence, defining $w \in L^1_{\text{loc}}(A)$ by

$$w(x) = \hat{u}(x) - v(|x|)$$

we see that w is radial and $\Delta^m w = 0$. Therefore $w \in C^\infty(A)$ and

$$w(x) = \hat{w}(x) = \bar{w}(|x|).$$

Since $\bar{u}(|x|) = \hat{u}(x) = \bar{w}(|x|) + v(|x|)$ we have $\bar{u} \in C^{2m-1}(a, b)$, $\bar{u}^{(2m-1)}$ is absolutely continuous on $[\alpha, \beta]$, and for almost all $r \in (a, b)$

$$\Delta^m_x \bar{u}(|x|)|_{|x|=r} = \Delta^m_x v(|x|)|_{|x|=r} = \overline{\Delta^m u}(r)$$

by (1.3.12). □

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Excerpt
More information

In what follows the function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(r) = \begin{cases} r^{2-n} & \text{if } n \geq 3; \\ \log \frac{5}{r} & \text{if } n = 2. \end{cases} \tag{1.3.13}$$

Lemma 1.14 *Suppose $v \in C^{2m-1}(0, 2R)$ and $v^{(2m-1)}$ is absolutely continuous on every closed subinterval of $(0, 2R)$.*

(i) If there exists a positive constant B such that

$$\Delta_x^m v(|x|)|_{|x|=r} \leq B\Gamma(r) \quad \textit{for almost all } r \in (0, 2R) \tag{1.3.14}$$

then there exists a positive constant C such that

$$v(r) \leq C\Gamma(r) \quad \textit{for all } r \in (0, R). \tag{1.3.15}$$

(ii) If v is nonnegative and for some nonnegative function $g \in L^1_{loc}(0, 2R)$,

$$-\Delta_x^m v(|x|)|_{|x|=r} = g(r) \quad \textit{for almost all } r \in (0, 2R) \tag{1.3.16}$$

then

$$\int_0^R r^{2m+n-3} g(r)\, dr < \infty. \tag{1.3.17}$$

Proof (i) We use induction. Suppose $m = 1$. Let $h(r) = \Delta_x v(|x|)|_{|x|=r}$. Then $h \in L^1_{\text{loc}}(0, 2R)$ and for $0 < r < R$ we have

$$v(r) = v(R) - v'(R) \int_r^R \left(\frac{R}{\rho}\right)^{n-1} d\rho + \int_r^R s^{1-n} \int_s^R h(\rho)\rho^{n-1}\, d\rho\, ds. \tag{1.3.18}$$

Since $h \leq B\Gamma$ almost everywhere in $(0, 2R)$ we have

$$\int_s^R h(\rho)\rho^{n-1}\, d\rho \leq B \int_0^R \Gamma(\rho)\rho^{n-1}\, d\rho < \infty \quad \text{for } s \in (0, R).$$

It therefore follows from (1.3.18) that there exists a positive constant C such that v satisfies (1.3.15).

Assume inductively that Lemma 1.14(i) is true with m replaced with $m - 1$, where $m \geq 2$, and v satisfies the conditions of Lemma 1.14(i). Let

$$w(r) = \Delta_x^{m-1} v(|x|)|_{|x|=r} \quad \text{for } 0 < r < 2R.$$

Then $w \in C^1(0, 2R)$ and w' is absolutely continuous on every closed subinterval of $(0, 2R)$. Since $w(|x|) = \Delta_x^{m-1} v(|x|)$ it follows from (1.3.14) that

$$\Delta_x w(|x|)|_{|x|=r} = \Delta_x^m v(|x|)|_{|x|=r} \leq B\Gamma(r)$$

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