# 1

# Semigroups of Operators and Cosine Operator Functions

In this introductory chapter, I present the rudiments of the theory of strongly continuous semigroups of operators and the related theory of cosine operator functions (cosine families). These theories, especially the first one, being well known, and given the number of books devoted to the subject, I felt it unnecessary to present all the involved details here, but appropriate to recall basic facts. Experts may simply skip this chapter, and students familiar with the theory may test their fluency by checking whether they remember how to prove the facts presented here (and perhaps do the nonstandard exercises at the end). All of the remaining readers are asked to consult one or several of [9, 49, 132, 101, 103, 118, 122, 127, 128, 129, 163, 180, 193, 201, 215, 256, 284, 324, 334, 343, 353] or other relevant monographs. The real newcomers may want to begin by reading [259]. A unified proof of the Hille–Yosida and Sova–Da Prato–Giusti Theorems is given in Appendix A.

## $C_0$ Semigroups

A $C_0$ semigroup or a strongly continuous semigroup in a Banach space $\mathbb{X}$ is a family $T = (T(t))_{t\geq 0}$ (written also as $\{T(t), t \geq 0\}$) of bounded linear operators such that:

(a) $T(t)T(s) = T(t+s), t, s \geq 0$,
(b) $T(0) = I_{\mathbb{X}}$,
(c) $\lim_{t\to 0} T(t)x = x, x \in \mathbb{X}$,

with the last limit in the strong topology in $\mathbb{X}$. If merely the first two conditions are satisfied the family is said to be a **semigroup**. The three properties together, combined with the Banach–Steinhaus uniform boundedness principle, imply that there exist $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq M e^{\omega t}$. Moreover, the map

$[0, \infty) \ni t \mapsto T(t)x$ is continuous (in the strong topology) for all $x \in \mathbb{X}$. Hence, the Laplace transform $R_\lambda = \int_0^\infty e^{-\lambda t} T(t)\, dt$ of the semigroup is well defined for all $\lambda > \omega$, as a strong improper Riemann integral, and we have $\|R_\lambda\| \le \frac{M}{\lambda - \omega}$. Using the semigroup property, we check that:

$$R_\lambda^n = \int_0^\infty \dots \int_0^\infty e^{-\lambda \sum_{i=1}^n t_i} T\left(\sum_{i=1}^n t_i\right) dt_1 \dots dt_n,$$

which gives $\|R_\lambda^n\| \le M(\lambda - \omega)^{-n}$, $n \ge 1$.

The infinitesimal generator of $(T(t))_{t\ge 0}$ is defined as:

$$Ax = \lim_{t\to 0} \frac{1}{t}(T(t)x - x),$$

for those $x$ for which the limit exists in the strong topology. Because all integrals of the form $\int_0^h T(t)x\, ds$, $x \in \mathbb{X}$, $h > 0$ belong to the domain $D(A)$ of $A$ and $\lim_{h\to 0} \frac{1}{h} \int_0^h T(t)x\, ds = x$, $A$ is densely defined. Furthermore, for $x \in D(A)$, $T(t)x = x + \int_0^t T(s)Ax\, ds$, and this implies that $A$ is closed. Then, $R_\lambda$ turns out to be the **resolvent** of $A$ in the sense that $\lambda - A$ has a bounded left and right inverse: (a) $R_\lambda(\lambda - A)x = x$ for $x \in D(A)$, and (b) $R_\lambda x$ belongs to $D(A)$ for all $x \in \mathbb{X}$ and $(\lambda - A)R_\lambda x = x$ (in particular, the range of $\lambda - A$ is the whole of $\mathbb{X}$; this is the all-important **range condition**). Therefore, we write $R_\lambda = (\lambda - A)^{-1}$. We note that the equation:

$$\lambda x - Ax = y, \tag{1.1}$$

where $y \in \mathbb{X}$ is given and $x \in D(A)$ is to be found, is called the **resolvent equation** for $A$.

This implies in turn that $R_\lambda$, $\lambda > 0$ satisfies the **Hilbert Equation**:

$$R_\lambda - R_\mu = (\mu - \lambda)R_\mu R_\lambda, \qquad \lambda, \mu > 0, \tag{1.2}$$

which could also be obtained from the fact that $R_\lambda$ is the Laplace transform of the semigroup.

The Hille–Yosida–Feller–Phillips–Miyadera Theorem states that the conditions on $A$ mentioned here are not only necessary but also sufficient for $A$ to be the generator of a strongly continuous semigroup.

**Theorem 1.1 (Hille–Yosida–Feller–Phillips–Miyadera)** *An operator $A$ is the generator of a strongly continuous semigroup $\{T(t), t \ge 0\}$ with $\|T(t)\| \le Me^{\omega t}$ iff:*

- *$A$ is closed and densely defined,*
- *for all $\lambda > \omega$, $\lambda - A$ has a continuous left and right inverse $(\lambda - A)^{-1}$,*
- *$\|(\lambda - A)^{-n}\| \le M(\lambda - \omega)^{-n}$, $n \ge 1$, $\lambda > \omega$.*

Using the Hilbert Equation and the estimate $\|(\lambda - A)^{-1}\| \le \frac{M}{\lambda-\omega}$, we may prove that the map $(\omega,\infty) \ni \lambda \to (\lambda - A)^{-1} \in \mathcal{L}(\mathbb{X})$ is differentiable with $\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda - A)^{-1} = -(\lambda - A)^{-2}$. Hence, by induction:

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda - A)^{-1} = (-1)^n n!(\lambda - A)^{-(n+1)}, \qquad n \ge 1$$

so that the third condition may be equivalently expressed as:

- $\|\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}(\lambda - A)^{-1}\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}$, $n \ge 0, \lambda > \omega$.

We note that $A$ generates a strongly continuous semigroup $\{T(t), t \ge 0\}$ iff $(\lambda - A)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\,\mathrm{d}t$; this shows in particular that an operator must not generate two different semigroups. We will write $(\mathrm{e}^{tA})_{t\ge 0}$, $\{\mathrm{e}^{tA}, t \ge 0\}$ or $\{T_A(t), t \ge 0\}$ for the semigroup generated by $A$.

The Phillips Perturbation Theorem says that if $A$ is the generator of a strongly continuous semigroup, then so is $A + B$ (with domain $\mathcal{D}(B) = \mathcal{D}(A)$) where $B$ is a bounded operator. Moreover,

$$\mathrm{e}^{t(A+B)} = \sum_{n=0}^{\infty} S_n(t), \text{ where}$$

$$S_0(t) = \mathrm{e}^{tA}, S_{n+1}(t) = \int_0^t \mathrm{e}^{(t-s)A} B S_n(s)\,\mathrm{d}s, \qquad n \ge 0.$$

## Cosine Operator Functions

A strongly continuous cosine operator function (or a cosine family) is a family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators satisfying:

(a) $2C(t)C(s) = C(t+s) + C(t-s), t, s \in \mathbb{R}$,
(b) $C(0) = I_{\mathbb{X}}$,
(c) $\lim_{t\to 0} C(t)x = x, x \in \mathbb{X}$.

As in the case of semigroups, there exist $M \ge 1$ and $\omega \ge 0$ (note the restriction on $\omega$; see Proposition 3.14.6 in [9] and our Exercise 61.2), such that $\|C(t)\| \le M\mathrm{e}^{\omega t}$. Also, $C(t) = C(-t)$ and $\{C(t), t \in \mathbb{R}\}$ is a strongly continuous family. Finally, its Laplace transform is uniquely determined by:

$$\lambda(\lambda^2 - A)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t} C(t)\,\mathrm{d}t, \lambda > \omega, \tag{1.3}$$

where $A$ is the infinitesimal generator defined as:

$$Ax = \lim_{t\to 0} \frac{2[C(t)x - x]}{t^2},$$

for those $x$ for which this limit exists. Again, $A$ is of necessity closed and densely defined. The analogue of the Hille–Yosida–Feller–Phillips–Miyadera Theorem (due to Sova [319], and Da Prato and Giusti [290]) is as follows.

**Theorem 1.2** *An operator $A$ is the generator of a strongly continuous cosine operator family $\{C(t), t \in \mathbb{R}\}$ such that $\|C(t)\| \le M e^{\omega t}$ iff:*

- *$A$ is closed and densely defined,*
- *for all $\lambda > \omega$, $\lambda^2 - A$ has a continuous left and right inverse $(\lambda^2 - A)^{-1}$,*
- $\| \frac{d^n}{d\lambda^n} [\lambda(\lambda^2 - A)^{-1}] \| \le M n! (\lambda - \omega)^{-(n+1)}$, $n \ge 0, \lambda > \omega$.

Formula (1.3) shows that the generator uniquely determines the cosine family. I will write $\{\mathrm{Cos}_A(t), t \in \mathbb{R}\}$ for the cosine family generated by $A$. I note that there is a cosine family equivalent of the Phillips Perturbation Theorem (Corollaries 3.14.10 and 3.14.13 in [9]).

## Semigroups and Cosines

As it was seen earlier, the theories of strongly continuous semigroups and of cosine families are quite analogous (see, however, [62] or Chapter 61). In Appendix A, I will show that both the Hille–Yosida–Feller–Phillips–Miyadera and the Sova–Da Prato–Giusti generation theorems may be obtained from the Hennig–Neubrander Representation Theorem for Laplace transform. Here, I discuss two other connections between these theories.

Suppose that $\{\mathrm{Cos}_A(t), t \in \mathbb{R}\}$ is a strongly continuous cosine family. Then the formula:

$$T(t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \mathrm{Cos}_A(s)\, ds, \qquad t > 0 \tag{1.4}$$

defines a strongly continuous semigroup whose generator turns out to be $A$. In other words, generators of cosine families are of necessity generators of semigroups, even of holomorphic semigroups (see Chapter 15). The converse, however, is not true. There are generators of holomorphic semigroups that are not generators of cosine families. Relation (1.4) is known as the **Weierstrass Formula** [9, 139] or **subordination principle** [292]; see [154] for probabilistic aspects of this relationship.

The **Kisyński Theorem** establishes another connection between semigroups and cosines, as follows [9, 213]: Let again $\{\mathrm{Cos}_A(t), t \in \mathbb{R}\}$ be a strongly continuous cosine family in a Banach space $\mathbb{X}$ and let $\mathbb{X}_{\mathrm{Kis}}$ be the subset of $x \in \mathbb{X}$ such that $t \mapsto \mathrm{Cos}_A(t)x$ is continuously differentiable. Then, $\mathbb{X}_{\mathrm{Kis}}$ is a Banach space when equipped with the norm $\|x\|_{\mathrm{Kis}} = \|x\| + \sup_{t \in [0,1]} \| \frac{dC_A(t)x}{dt} \|$, and the operator $\mathcal{A}(y,x) = (x, Ay)$ with domain $\mathcal{D}(\mathcal{A})$ composed of $(y,x)$ with $y \in \mathcal{D}(A)$, $x \in \mathbb{X}_{\mathrm{Kis}}$ generates a $C_0$ group in $\mathbb{X}_{\mathrm{Kis}} \times \mathbb{X}$ (i.e., both $\mathcal{A}$ and $-\mathcal{A}$ generate $C_0$ semigroups there), and we have:

$$\mathrm{e}^{t\mathcal{A}} = \begin{pmatrix} \mathrm{Cos}_A(t) & \int_0^t \mathrm{Cos}_A(s)\,\mathrm{d}s \\ \frac{\mathrm{dCos}_A(t)}{\mathrm{d}t} & \mathrm{Cos}_A(t) \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{1.5}$$

The converse is also true: if $A$ is a linear operator in a Banach space $\mathbb{X}$, if $\mathbb{X}_1$ is a Banach space such that:

$$\mathcal{D}(A) \subset \mathbb{X}_1 \subset \mathbb{X}, \tag{1.6}$$

and the topology in $\mathcal{D}(A)$ is not weaker than the topology in $\mathbb{X}_1$ and the topology in $\mathbb{X}_1$ is not weaker than the topology in $\mathbb{X}$ and $\mathcal{A}$ given here generates a $C_0$ group in $\mathbb{X}_1 \times \mathbb{X}$, then $A$ generates a strongly continuous cosine family in $\mathbb{X}$. Moreover, $\mathbb{X}_1$ is isomorphic to $\mathbb{X}_{\mathrm{Kis}}$. The $\mathbb{X}_{\mathrm{Kis}}$ is often referred to as the **Kisyński space**. Interestingly, if condition (1.6) is omitted, $\mathbb{X}_{\mathrm{Kis}}$ is not uniquely determined; see [237].

**CONVENTION** Throughout this book, when speaking about abstract Banach spaces, I use $x, y, z$, and so on to denote their elements. However, when dealing with applications, which usually involve function spaces, for their elements I write $f, g, h$ or $\phi, \psi$, and so on and use $x, y$, and $z$ to denote arguments of functions.

**Exercise 1.1** Let $A$ be the generator of a strongly continuous semigroup in a Banach space $\mathbb{X}$. Show that $\lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1}x = x, x \in \mathbb{X}$ and that $\lim_{\lambda\to\infty} \lambda[\lambda(\lambda - A)^{-1}x - x] = Ax, x \in D(A)$. In fact, the latter condition holds iff $x \in D(A)$.

**Exercise 1.2** Check that a bounded operator $A$ generates the semigroup (actually, a group) given by $\mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$ and the cosine family given by $\mathrm{Cos}_A(t) = \sum_{n=0}^{\infty} \frac{A^{2n}t^{2n}}{(2n)!}$. Verify the Weierstrass Formula in this case.

**Exercise 1.3** Let $\mathbb{X}$ be a Banach space, $z \in \mathbb{X}$ be a fixed element, and $f \in \mathbb{X}^*$ be a fixed linear functional. Consider the bounded linear operator $Ax = f(x)z$. Prove that:

$$\mathrm{e}^{tA}x = \begin{cases} x + \frac{f(x)}{r}(\mathrm{e}^{rt} - 1)z, & r \neq 0, \\ x + tf(x)z, & r = 0, \end{cases} \qquad t \geq 0, x \in \mathbb{X},$$

where $r = f(z)$.

**Exercise 1.4** Let $\mathbb{X}$ be a Banach space.

(a) Let $A \in \mathcal{L}(\mathbb{X})$ be a bounded linear operator satisfying $A^2 = A$. Show that $e^{tA} = I_{\mathbb{X}} + (e^t - 1)A$.
(b) Let $A \in \mathcal{L}(\mathbb{X})$ be a bounded linear operator satisfying $A^2 = 0$. Show that $e^{tA} = I_{\mathbb{X}} + tA$.
(c) Let $A \in \mathcal{L}(\mathbb{X})$ be a bounded linear operator satisfying $A^2 = -I_{\mathbb{X}}$. Show that $e^{tA} = (\cos t) I_{\mathbb{X}} + (\sin t) A$.
(d) Let $A \in \mathcal{L}(\mathbb{X})$ be a bounded linear operator satisfying $A^2 = I_{\mathbb{X}}$. Show that $e^{tA} = (\cosh t) I_{\mathbb{X}} + (\sinh t) A$.

**Exercise 1.5** Let $\mathbb{X} = C_0[0, \infty)$ be the space of continuous functions on $\mathbb{R}^+$, vanishing at infinity, and let $\mu$ be a non-negative, continuous function on $\mathbb{R}^+$ such that $\lim_{x\to\infty} \mu(x)$ exists (but may be infinite).

(a) Show that the related, possibly unbounded, operator $M : f \mapsto -\mu f$ (with domain composed of $f$ such that $\mu f \in \mathbb{X}$) is the generator of the contraction semigroup in $\mathbb{X}$ given by:

$$e^{Mt} f(x) = e^{-\mu(x)t} f(x), \qquad x, t \ge 0. \tag{1.7}$$

In the case in which $\lim_{x\to\infty} \mu(x) = \infty$, check that formula (1.7) does not define a strongly continuous semigroup in the space $C[0, \infty]$ of continuous functions with (finite) limits at $\infty$. *Hint:* take $f \equiv 1$.

(b) Let $\mu$ be positive. For $b \in \mathbb{X}$, let a bounded linear operator $B$ in $\mathbb{X}$ be defined by $Bf(x) = f(0)b(x)$, $x \in \mathbb{R}^+$. By (a) and the Phillips Perturbation Theorem, $M + B$ is the generator of a strongly continuous semigroup. Under the simplifying assumption that $\mu(0) = b(0)$, check directly that:

$$e^{t(M+B)} f(x) = e^{-\mu(x)t}(f(x) - f(0)q(x)) + f(0)q(x), \tag{1.8}$$

where $q(x) = \frac{b(x)}{\mu(x)}$ so that $q \in \mathbb{X}$. ($M + B$ is one of the terms in the predual of the generator of the McKendrick semigroup; see Chapter 50. See also Exercise 19.3.) *Hint:* Note that $q \in D(M)$; a direct calculation shows that the generator of the semigroup defined by the right-hand side of (1.8) extends $M + B$.

**Exercise 1.6**

(a) Let $\mathbb{X} = C[0, \infty]$ be the space of continuous functions on $[0, \infty)$ with limits at infinity, equipped with the usual supremum norm. Check that:

$$T(t)f(x) = f(x + t)$$

defines a strongly continuous semigroup of operators in $\mathbb{X}$, that the domain of its generator $A$ is composed of continuously differentiable functions $f$ such that $f' \in \mathbb{X}$ (deduce that $\lim_{x\to\infty} f'(x) = 0$), and that $Af = f'$.

(b) Let $\mathbb{X}$ be the space $C[0,1]$ of continuous functions on $[0,1]$, equipped with the usual supremum norm. Check that:

$$S(t)f(x) = f(xe^{-t})$$

defines a strongly continuous semigroup of operators in $\mathbb{X}$, that the domain of its generator $B$ is composed of functions $f$ such that $f'(x)$ exists for all $x \in (0,1]$ and $\lim_{x\to 0} xf'(x) = 0$. Also, $Bf(x) = -xf'(x)$, $x \in (0,1]$ (and clearly $Bf(0) = 0$).

(c) Check that:

$$U(t)f(x) = f(1-(1-x)e^{-t})$$

defines a strongly continuous semigroup of operators in the same Banach space $C[0,1]$, that the domain of its generator $C$ is composed of functions $f$ such that $f'(x)$ exists for all $x \in [0,1)$ and $\lim_{x\to 1}(1-x)f'(x) = 0$. Also, $Cf(x) = (1-x)f'(x)$, $x \in [0,1)$ (and clearly $Cf(1) = 1$).

**Exercise 1.7** Two semigroups $(e^{tA})_{t\ge 0}$ and $(e^{tB})_{t\ge 0}$ defined in Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively, are said to be isomorphic (or similar) iff there is an isomorphism $I : \mathbb{X} \to \mathbb{Y}$ such that:

$$Ie^{tA} = e^{tB}I, \qquad t \ge 0.$$

(a) Check that $x \in D(A)$ iff $Ix \in D(B)$ and that $IAx = BIx$.

(b) Check that all three semigroups of the previous exercise are similar and use this fact to find an independent proof of the characterizations of generators of semigroups in points (b) and (c) in the previous exercise.

**Exercise 1.8** Let $L^1(\mathbb{R}^+)$ be the space of (equivalence classes) of absolutely integrable functions on $\mathbb{R}^+$.

(a) Let $\{T(t), t \ge 0\}$ be the semigroup of shifts to the right:

$$T(t)\phi(x) = \begin{cases} \phi(x-t), & x \ge t \\ 0, & x < t, \end{cases} \qquad \phi \in L^1(\mathbb{R}^+).$$

Show that the domain of its generator is composed of absolutely continuous functions $\phi$ with $\phi(0) = 0$, i.e. such that $\phi(x) = \int_0^x \psi(y)\,dy$, $x \ge 0$ for some $\psi \in L^1(\mathbb{R}^+)$, and that $A\phi = -\phi' = -\psi$.

(b) Let $\mu \ge 0$ be a bounded, integrable function on $\mathbb{R}^+$. Show that $I \in \mathcal{L}(L^1(\mathbb{R}^+))$ given by $(I\phi)(x) = e^{\int_0^x \mu(y)\,dy}\phi(x)$ is an isomorphism of $L^1(\mathbb{R}^+)$

(with $\|I\| \le e^{\int_0^\infty \mu(x)\,dx}$ and $\|I^{-1}\| \le 1$) and check that the isomorphic image $T_\mu(t) = I^{-1}T(t)I$ of the semigroup of point (a) via $I$ is

$$T_\mu(t)\phi(x) = \begin{cases} e^{-\int_{x-t}^{x}\mu(y)\,dy}\phi(x-t), & x \ge t \\ 0, & x < t, \end{cases} \qquad \phi \in L^1(\mathbb{R}^+). \tag{1.9}$$

(c) Let $A_\mu$ be the generator of this semigroup. Using the fact that the product of two absolutely continuous functions is absolutely continuous, show that $D(A_\mu) = D(A)$, and that $A_\mu\phi = A\phi - \mu\phi$, $\phi \in D(A_\mu)$. (See also Exercise 2.3.)

**Exercise 1.9** Let $A$ be the operator in $l^2$, the space of square-summable sequences $(x_n)_{n\ge1}$, given by $D(A) = \{(\xi_n)_{n\ge1} \in l^1, (n\xi_n)_{n\ge1} \in l^2\}$, $A(\xi_n)_{n\ge1} = -(n\xi_n)_{n\ge1}$.

(a) Show that $A$ is a generator and that:

$$e^{tA}(\xi_n)_{n\ge1} = \left(e^{-nt}\xi_n\right)_{n\ge1}.$$

(b) Let $L^2(0,\pi)$ be the space of (classes of) square integrable functions on $(0,2\pi)$. Check that $e_n(x) = \sqrt{\frac{\pi}{2}}\sin x$, $x \in (0,\pi)$, $n \ge 1$, is a complete, orthonormal system (see, e.g., [107]), and conclude that $L^2(0,\pi)$ is isometrically isomorphic to $l^2$ with isomorphism $I : L^2(0,\pi) \to l^2$ given by:

$$If = ((f, e_n))_{n\ge1},$$

where the inner parenthesis denotes the scalar product in $L^2(0,\pi)$.

(c) Let $B$ be the operator in $L^2(0,\pi)$ with domain $D(B)$ composed of functions $f$ that can be represented in the form:

$$f(x) = ax + \int_0^x \int_0^y h(z)\,dz\,dy$$

where $a = -\frac{1}{\pi}\int_0^\pi \int_0^y h(z)\,dz\,dy$, for some $h \in L^2(0,\pi)$, and given by:

$$Af = h.$$

Use (a) and (b) to show that $B$ is a generator, and find the form of $e^{tB}$.

**Exercise 1.10** Let $BUC(\mathbb{R})$ be the space of uniformly continuous functions on the real line, equipped with the supremum norm. Check that:

$$C(t)f(x) = \frac{1}{2}[f(x+t) + f(x-t)], \qquad t, x \in \mathbb{R},$$

defines a strongly continuous cosine family. Prove that its generator is given by $Af = f''$ with domain composed of twice continuously differentiable members of $BUC(\mathbb{R})$ such that $f'' \in BUC(\mathbb{R})$.

**Exercise 1.11** Let $BUC(\mathbb{R}^+)$ be the space of uniformly continuous functions on the half-line $\mathbb{R}^+$ equipped with the supremum norm. For $f \in BUC(\mathbb{R}^+)$, let $f_e \in BUC(\mathbb{R})$ be its even extension to the whole line. Check that:

$$C(t)f(x) = \frac{1}{2}[f_e(x+t) + f_e(x-t)], \qquad t \in \mathbb{R}, x \in \mathbb{R}^+,$$

defines a strongly continuous cosine family. Prove that its generator is given by $Af = f''$ with domain composed of twice continuously differentiable members of $BUC(\mathbb{R}^+)$ such that $f'' \in BUC(\mathbb{R}^+)$, and $f'(0) = 0$.

**Exercise 1.12** Let $C_\mathrm{p}[0,1]$ be the space of continuous functions $f$ on $[0,1]$ such that $f(0) = f(1)$. Let $f_\mathrm{p}$ be the periodic extension of a $f \in C_\mathrm{p}[0,1]$ given by $f_\mathrm{p}(x+k) = f(x)$, $x \in [0,1]$ where $k$ is any integer. Show that:

$$C(t)f(x) = \frac{1}{2}[f_\mathrm{p}(x+t) + f_\mathrm{p}(x-t)], \qquad t \in \mathbb{R}, x \in \mathbb{R}^+,$$

defines a strongly continuous cosine family and prove that its generator is $Af = f''$ with domain composed of twice continuously differentiable members of $C_\mathrm{p}[0,1]$ such that $f', f'' \in C_\mathrm{p}[0,1]$.

**Exercise 1.13** Let $C[-\infty,\infty]$ be the space of continuous functions on $\mathbb{R}$ with limits at $+\infty$ and $-\infty$, and let $\{T(t), t \ge 0\}$ be operators defined by:

$$T(t)f(x) = \begin{cases} f(x+\sigma_1 t), & x \in [0,\infty), \\ f(\frac{\sigma_1}{\sigma_{-1}}x + \sigma_1 t), & x \in (-\sigma_{-1}t, 0), \\ f(x+\sigma_{-1}t), & x \in (-\infty, -\sigma_{-1}t], \end{cases}$$

where $\sigma_1$ and $\sigma_{-1}$ are given positive constants.

(a) Show that this is a strongly continuous semigroup with generator $A$ given by:

$$Af = \begin{cases} \sigma_1 f'(x), & x \ge 0 \\ \sigma_{-1} f'(x), & x < 0, \end{cases}$$

defined for all $f$ satisfying the following conditions: (i) $f$ is continuously differentiable for $x > 0$, with the derivative having limits at $+\infty$ and 0 – the latter limit is denoted $f'(0+)$. (ii) $f$ is continuously differentiable for $x < 0$ with derivative having limits at $-\infty$ and 0 – the latter limit is denoted $f'(0-)$. (iii) We have $\sigma_{-1}f'(0-) = \sigma_1 f'(0+)$.

(b) Let $\alpha(x) = \sigma_{\operatorname{sgn} x}x$. Check that $I : f \mapsto f \circ \alpha$ is an isometric isomorphism of $C[-\infty,\infty]$. Redo point (a) by noting that $S(t) = IT(t)I^{-1}$ is the translation to the left: $S(t)f(x) = f(x+t)$, $x \in \mathbb{R}$, $t \ge 0$.

(c) Prove that the semigroup may be extended to a group $\{T(t), t \in \mathbb{R}\}$, and find an explicit formula for $T(t), t < 0$.
(d) Prove that $C(t) = \frac{1}{2}(T(t) + T(-t)), t \in \mathbb{R}$ defines a cosine family and find its generator.