

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 155

Editorial Board

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

MARTINGALES IN BANACH SPACES

This book is focused on the major applications of martingales to the geometry of Banach spaces, but a substantial discussion of harmonic analysis in Banach space valued Hardy spaces is presented. Exciting links between super-reflexivity and some metric spaces related to computer science are covered, as is an outline of the recently developed theory of non-commutative martingales, which has natural connections with quantum physics and quantum information theory.

Requiring few prerequisites and providing fully detailed proofs for the main results, this self-contained study is accessible to graduate students with basic knowledge of real and complex analysis and functional analysis. Chapters can be read independently, each building from introductory notes, and the diversity of topics included also means this book can serve as the basis for a variety of graduate courses.

Gilles Pisier was a professor at the University of Paris VI from 1981 to 2010 and has been Emeritus Professor since then. He has been a distinguished professor and holder of the Owen Chair in Mathematics at Texas A&M University since 1985. His international prizes include the Salem Prize in harmonic analysis (1979), the Ostrowski Prize (1997), and the Stefan Banach Medal (2001). He is a member of the Paris Académie des sciences, a Foreign Member of the Polish and Indian Academies of Science, and a Fellow of both the IMS and the AMS. He is also the author of several books, notably *The Volume of Convex Bodies and Banach Space Geometry* (1989) and *Introduction to Operator Space Theory* (2002), both published by Cambridge University Press.

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.
For a complete series listing, visit: www.cambridge.org/mathematics.

Already published

- 116 D. Applebaum *Lévy processes and stochastic calculus (2nd Edition)*
- 117 T. Szamuely *Galois groups and fundamental groups*
- 118 G. W. Anderson, A. Guionnet & O. Zeitouni *An introduction to random matrices*
- 119 C. Perez-Garcia & W. H. Schikhof *Locally convex spaces over non-Archimedean valued fields*
- 120 P. K. Friz & N. B. Victoir *Multidimensional stochastic processes as rough paths*
- 121 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Representation theory of the symmetric groups*
- 122 S. Kalikow & R. McCutcheon *An outline of ergodic theory*
- 123 G. F. Lawler & V. Limic *Random walk: A modern introduction*
- 124 K. Lux & H. Pahlings *Representations of groups*
- 125 K. S. Kedlaya *p -adic differential equations*
- 126 R. Beals & R. Wong *Special functions*
- 127 E. de Faria & W. de Melo *Mathematical aspects of quantum field theory*
- 128 A. Terras *Zeta functions of graphs*
- 129 D. Goldfeld & J. Hundley *Automorphic representations and L -functions for the general linear group, I*
- 130 D. Goldfeld & J. Hundley *Automorphic representations and L -functions for the general linear group, II*
- 131 D. A. Craven *The theory of fusion systems*
- 132 J. Väänänen *Models and games*
- 133 G. Malle & D. Testerman *Linear algebraic groups and finite groups of Lie type*
- 134 P. Li *Geometric analysis*
- 135 F. Maggi *Sets of finite perimeter and geometric variational problems*
- 136 M. Brodmann & R. Y. Sharp *Local cohomology (2nd Edition)*
- 137 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, I*
- 138 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, II*
- 139 B. Helffer *Spectral theory and its applications*
- 140 R. Pemantle & M. C. Wilson *Analytic combinatorics in several variables*
- 141 B. Branner & N. Fagella *Quasiconformal surgery in holomorphic dynamics*
- 142 R. M. Dudley *Uniform central limit theorems (2nd Edition)*
- 143 T. Leinster *Basic category theory*
- 144 I. Arzhantsev, U. Derenthal, J. Hausen & A. Laface *Cox rings*
- 145 M. Viana *Lectures on Lyapunov exponents*
- 146 J.-H. Evertse & K. Györy *Unit equations in Diophantine number theory*
- 147 A. Prasad *Representation theory*
- 148 S. R. Garcia, J. Mashreghi & W. T. Ross *Introduction to model spaces and their operators*
- 149 C. Godsil & K. Meagher *Erdős–Ko–Rado theorems: Algebraic approaches*
- 150 P. Mattila *Fourier analysis and Hausdorff dimension*
- 151 M. Viana & K. Oliveira *Foundations of ergodic theory*
- 152 V. I. Paulsen & M. Raghupathi *An introduction to the theory of reproducing kernel Hilbert spaces*
- 153 R. Beals & R. Wong *Special functions and orthogonal polynomials (2nd Edition)*
- 154 V. Jurdjevic *Optimal control and geometry: Integrable systems*
- 155 G. Pisier *Martingales in Banach Spaces*

Martingales in Banach Spaces

GILLES PISIER
Texas A&M University



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press & Assessment
978-1-107-13724-0 — Martingales in Banach Spaces
Gilles Pisier
Frontmatter
[More Information](#)



CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107137240

© Gilles Pisier 2016

This publication is in copyright. Subject to statutory exception and to the provisions
of relevant collective licensing agreements, no reproduction of any part may take
place without the written permission of Cambridge University Press & Assessment.

First published 2016

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-13724-0 Hardback

Cambridge University Press & Assessment has no responsibility for the persistence
or accuracy of URLs for external or third-party internet websites referred to in this
publication and does not guarantee that any content on such websites is, or will
remain, accurate or appropriate.

Contents

| | |
|---|---------------|
| <i>Introduction</i> | <i>page</i> x |
| <i>Description of the contents</i> | xiv |
| 1 Banach space valued martingales | 1 |
| 1.1 Banach space valued L_p -spaces | 1 |
| 1.2 Banach space valued conditional expectation | 7 |
| 1.3 Martingales: basic properties | 9 |
| 1.4 Examples of filtrations | 12 |
| 1.5 Stopping times | 17 |
| 1.6 Almost sure convergence: Maximal inequalities | 19 |
| 1.7 Independent increments | 28 |
| 1.8 Phillips's theorem | 31 |
| 1.9 Reverse martingales | 34 |
| 1.10 Continuous time* | 36 |
| 1.11 Notes and remarks | 41 |
| 2 Radon-Nikodým property | 42 |
| 2.1 Vector measures | 42 |
| 2.2 Martingales, dentability and the Radon-Nikodým property | 46 |
| 2.3 The dual of $L_p(B)$ | 57 |
| 2.4 Generalizations of $L_p(B)$ | 60 |
| 2.5 The Krein-Milman property | 63 |
| 2.6 Edgar's Choquet theorem | 67 |
| 2.7 The Lewis-Stegall theorem | 70 |
| 2.8 Notes and remarks | 73 |

| | | |
|----------|---|-----|
| 3 | Harmonic functions and RNP | 76 |
| 3.1 | Harmonicity and the Poisson kernel | 76 |
| 3.2 | The h^p spaces of harmonic functions on D | 80 |
| 3.3 | Non-tangential maximal inequalities: boundary behaviour | 87 |
| 3.4 | Harmonic functions and RNP | 97 |
| 3.5 | Brownian martingales* | 101 |
| 3.6 | Notes and remarks | 110 |
| 4 | Analytic functions and ARNP | 112 |
| 4.1 | Subharmonic functions | 112 |
| 4.2 | Outer functions and $H^p(D)$ | 116 |
| 4.3 | Banach space valued H^p -spaces for $0 < p \leq \infty$ | 118 |
| 4.4 | Analytic Radon-Nikodým property | 127 |
| 4.5 | Hardy martingales and Brownian motion* | 131 |
| 4.6 | B -valued h^p and H^p over the half-plane U^* | 140 |
| 4.7 | Further complements* | 146 |
| 4.8 | Notes and remarks | 148 |
| 5 | The UMD property for Banach spaces | 151 |
| 5.1 | Martingale transforms (scalar case): Burkholder's inequalities | 151 |
| 5.2 | Square functions for B -valued martingales: Kahane's inequalities | 154 |
| 5.3 | Definition of UMD | 159 |
| 5.4 | Gundy's decomposition | 162 |
| 5.5 | Extrapolation | 168 |
| 5.6 | The UMD_1 property: Burgess Davis decomposition | 175 |
| 5.7 | Examples: UMD implies super-RNP | 180 |
| 5.8 | Dyadic UMD implies UMD | 183 |
| 5.9 | The Burkholder-Rosenthal inequality | 190 |
| 5.10 | Stein inequalities in UMD spaces | 195 |
| 5.11 | Burkholder's geometric characterization of UMD space | 197 |
| 5.12 | Appendix: hypercontractivity on $\{-1, 1\}$ | 206 |
| 5.13 | Appendix: Hölder-Minkowski inequality | 207 |
| 5.14 | Appendix: basic facts on weak- L_p | 209 |
| 5.15 | Appendix: reverse Hölder principle | 210 |
| 5.16 | Appendix: Marcinkiewicz theorem | 212 |
| 5.17 | Appendix: exponential inequalities and growth of L_p -norms | 214 |
| 5.18 | Notes and remarks | 214 |

| | | |
|-----------|---|-----|
| 6 | The Hilbert transform and UMD Banach spaces | 218 |
| 6.1 | Hilbert transform: HT spaces | 218 |
| 6.2 | Bourgain's transference theorem: HT implies UMD | 228 |
| 6.3 | UMD implies HT | 233 |
| 6.4 | UMD implies HT (with stochastic integrals)* | 246 |
| 6.5 | Littlewood-Paley inequalities in UMD spaces | 249 |
| 6.6 | The Walsh system Hilbert transform | 254 |
| 6.7 | Analytic UMD property* | 255 |
| 6.8 | UMD operators* | 257 |
| 6.9 | Notes and remarks | 260 |
| 7 | Banach space valued H^1 and BMO | 263 |
| 7.1 | Banach space valued H^1 and BMO: Fefferman's duality theorem | 263 |
| 7.2 | Atomic B -valued H^1 | 266 |
| 7.3 | H^1 , BMO and atoms for martingales | 282 |
| 7.4 | Regular filtrations | 291 |
| 7.5 | From dyadic BMO to classical BMO | 292 |
| 7.6 | Notes and remarks | 297 |
| 8 | Interpolation methods (complex and real) | 299 |
| 8.1 | The unit strip | 300 |
| 8.2 | The complex interpolation method | 303 |
| 8.3 | Duality for the complex method | 321 |
| 8.4 | The real interpolation method | 332 |
| 8.5 | Real interpolation between L_p -spaces | 336 |
| 8.6 | The K -functional for $(L_1(B_0), L_\infty(B_1))$ | 342 |
| 8.7 | Real interpolation between vector valued L_p -spaces | 346 |
| 8.8 | Duality for the real method | 352 |
| 8.9 | Reiteration for the real method | 357 |
| 8.10 | Comparing the real and complex methods | 362 |
| 8.11 | Symmetric and self-dual interpolation pairs | 363 |
| 8.12 | Notes and remarks | 368 |
| 9 | The strong p-variation of scalar valued martingales | 372 |
| 9.1 | Notes and remarks | 380 |
| 10 | Uniformly convex Banach space valued martingales | 382 |
| 10.1 | Uniform convexity | 382 |
| 10.2 | Uniform smoothness | 395 |
| 10.3 | Uniform convexity and smoothness of L_p | 404 |
| 10.4 | Type, cotype and UMD | 406 |

| | | |
|-----------|---|-----|
| viii | <i>Contents</i> | |
| | 10.5 Square function inequalities in q -uniformly convex and p -uniformly smooth spaces | 417 |
| | 10.6 Strong p -variation, uniform convexity and smoothness | 423 |
| | 10.7 Notes and remarks | 425 |
| 11 | Super-reflexivity | 428 |
| | 11.1 Finite representability and super-properties | 428 |
| | 11.2 Super-reflexivity and inequalities for basic sequences | 433 |
| | 11.3 Uniformly non-square and J -convex spaces | 446 |
| | 11.4 Super-reflexivity and uniform convexity | 454 |
| | 11.5 Strong law of large numbers and super-reflexivity | 457 |
| | 11.6 Complex interpolation: θ -Hilbertian spaces | 459 |
| | 11.7 Complex analogues of uniform convexity* | 463 |
| | 11.8 Appendix: ultrafilters, ultraproducts | 472 |
| | 11.9 Notes and remarks | 474 |
| 12 | Interpolation between strong p-variation spaces | 477 |
| | 12.1 The spaces $v_p(B)$, $\mathcal{W}_p(B)$ and $\mathcal{W}_{p,q}(B)$ | 478 |
| | 12.2 Duality and quasi-reflexivity | 482 |
| | 12.3 The intermediate spaces $u_p(B)$ and $v_p(B)$ | 485 |
| | 12.4 L_q -spaces with values in v_p and \mathcal{W}_p | 489 |
| | 12.5 Some applications | 492 |
| | 12.6 K -functional for $(v_r(B), \ell_\infty(B))$ | 494 |
| | 12.7 Strong p -variation in approximation theory | 496 |
| | 12.8 Notes and remarks | 499 |
| 13 | Martingales and metric spaces | 500 |
| | 13.1 Exponential inequalities | 500 |
| | 13.2 Concentration of measure | 503 |
| | 13.3 Metric characterization of super-reflexivity: trees | 506 |
| | 13.4 Another metric characterization of super-reflexivity: diamonds | 512 |
| | 13.5 Markov type p and uniform smoothness | 519 |
| | 13.6 Notes and remarks | 521 |
| 14 | An invitation to martingales in non-commutative L_p-spaces* | 523 |
| | 14.1 Non-commutative probability space | 523 |
| | 14.2 Non-commutative L_p -spaces | 524 |
| | 14.3 Conditional expectations: non-commutative martingales | 527 |
| | 14.4 Examples | 529 |

| | | |
|-------|---|-----|
| | <i>Contents</i> | ix |
| 14.5 | Non-commutative Khintchin inequalities | 531 |
| 14.6 | Non-commutative Burkholder inequalities | 533 |
| 14.7 | Non-commutative martingale transforms | 535 |
| 14.8 | Non-commutative maximal inequalities | 537 |
| 14.9 | Martingales in operator spaces | 539 |
| 14.10 | Notes and remarks | 541 |
| | <i>Bibliography</i> | 542 |
| | <i>Index</i> | 560 |

Introduction

Martingales (with discrete time) lie at the centre of this book. They are known to have major applications to virtually every corner of probability theory. Our central theme is their applications to the geometry of Banach spaces.

We should emphasize that we do *not* assume any knowledge about scalar valued martingales. Actually, the beginning of this book gives a self-contained introduction to the basic martingale convergence theorems for which the use of the norm of a vector valued random variable instead of the modulus of a scalar one makes little difference. Only when we consider the ‘boundedness implies convergence’ phenomenon does it start to matter. Indeed, this requires the Banach space B to have the Radon-Nikodým property (RNP). But even at this point, the reader who wishes to concentrate on the scalar case could simply assume that B is finite-dimensional and disregard all the infinite-dimensional technical points. The structure of the proofs remains pertinent if one does so. In fact, it may be good advice for a beginner to do a first reading in this way. One could argue similarly about the property of ‘unconditionality of martingale differences’ (UMD): although perhaps the presence of a Banach space norm is more disturbing there, our reader could assume at first reading that B is a Hilbert space, thus getting rid of a number of technicalities to which one can return later.

A major feature of the UMD property is its equivalence to the boundedness of the Hilbert transform (HT). Thus we include a substantial excursion in (Banach space valued) harmonic analysis to explain this.

Actually, connections with harmonic analysis abound in this book, as we include a rather detailed exposition of the boundary behaviour of B -valued harmonic (resp. analytic) functions in connections with the RNP (resp. analytic RNP) of the Banach space B . We introduce the corresponding B -valued Hardy spaces in analogy with their probabilistic counterparts. We are partly motivated

by the important role they play in operator theory, when one takes for B the space of bounded operators (or the Schatten p -class) on a Hilbert space.

Hardy spaces are closely linked with martingales via Brownian motion: indeed, for any B -valued bounded harmonic (resp. analytic) function u on the unit disc D , the composition $(u(W_{t \wedge T}))_{t > 0}$ of u with Brownian motion stopped before it exits D is an example of a continuous B -valued martingale, and its boundary behaviour depends in general on whether B has the RNP (resp. analytic RNP). We describe this connection with Brownian motion in detail, but we refrain from going too far on that road, remaining faithful to our discrete time emphasis. However, we include short sections summarizing just what is necessary to understand the connections with Brownian martingales in the Banach valued context, together with pointers to the relevant literature. In general, the sections that are a bit far off our main goals are marked by an asterisk. For instance, we describe in §7.1 the Banach space valued version of Fefferman's duality theorem between H^1 and BMO . While this is not really part of martingale theory, the interplay with martingales, both historically and heuristically, is so obvious that we felt we had to include it. The asterisked sections could be kept for a second reading.

In addition to the RN and UMD properties, our third main theme is super-reflexivity and its connections with uniform convexity and smoothness. Roughly, we relate the geometric properties of a Banach space B with the study of the p -variation

$$S_p(f) = \left(\sum_1^\infty \|f_n - f_{n-1}\|_B^p \right)^{1/p}$$

of B -valued martingales (f_n) . Depending on whether $S_p(f) \in L_p$ is necessary or sufficient for the convergence of (f_n) in $L_p(B)$, we can find an equivalent norm on B with modulus of uniform convexity (resp. smoothness) 'at least as good as' the function $t \rightarrow t^p$.

We also consider the strong p -variation

$$V_p(f) = \sup_{0=n(0) < n(1) < n(2) < \dots} \left(\sum_1^\infty \|f_{n(k)} - f_{n(k-1)}\|_B^p \right)^{1/p}$$

of a martingale. For that topic (exceptionally) we devote an entire chapter only to the scalar case. Our crucial tool here is the 'real interpolation method'. Real and complex interpolation in general play an important role in L_p -space theory, so we find it natural to devote a significant amount of space to these two 'methods'.

We allow ourselves several excursions aiming to illustrate the efficiency of martingales, for instance to the concentration of measure phenomenon. We also

describe some exciting recent work on non-linear properties of metric spaces analogous to uniform convexity/smoothness and type for metric spaces.

We originally intended to include in this book a detailed presentation of ‘non-commutative’ martingale theory, but that part became so big that we decided to make it the subject of a (hopefully forthcoming) separate volume to be published, perhaps on the author’s web page. We merely outline its contents in the last chapter, devoted to non-commutative L_p -spaces. There the complex interpolation method becomes a central tool.

The book should be accessible to graduate students, requiring only the basics of real and complex analysis (mainly Lebesgue integration) and basic functional analysis (mainly duality, the weak and strong topologies and reflexivity of Banach spaces). Our choice is to give fully detailed proofs for the main results and to indicate references to the refinements in the ‘Notes and Remarks’ or the asterisked sections. We strive to make the presentations self-contained, and when given a choice, we opt for simplicity over maximal generality. For instance, we restrict the Banach space valued harmonic analysis to functions with domains in the unit disc D or the upper half-plane U in \mathbb{C} (or their boundary $\partial D = \mathbb{T}$ or $\partial U = \mathbb{R}$). We feel the main ideas are easier to grasp in the real or complex uni-dimensional case.

The topics (martingales, H^p -space theory, interpolation, Banach space geometry) are quite diverse and should appeal to several distinct audiences. The main novelty is the choice to bring all these topics together in the various parts of this single volume. We should emphasize that the different parts can be read independently, and each time their start is introductory.

There are natural groupings of chapters, such as 1-2-10-11 or 3-4-5-6 (possibly including parts of 1 and 2, but not necessarily), which could form the basis for a graduate course.

Depending on his or her background, a reader is likely to choose to concentrate on different parts. We hope probabilist graduate students will benefit from the detailed introductory presentation of basic H^p -space theory, its connections with martingales, the links with Banach space geometry and the detailed treatment of interpolation theory (which we illustrate by applications to the strong p -variation of martingales), while graduate students with interest in functional analysis and Banach spaces should benefit more from the initial detailed presentation of basic martingale theory. In addition, we hope to attract readers with interest in computer science wishing to see the sources of the various recent developments on finite metric spaces described in Chapter 13. A reader with an advanced knowledge of harmonic analysis and H^p -theory will probably choose to skip the introductory part on that direction, which is written

with non-specialists in mind, and concentrate on the issues specific to Banach space valued functions related to the UMD property and the Hilbert transform.

The choice to include so much background on the real and complex interpolation methods in Chapter 8 is motivated by its crucial importance in Banach space valued L_p -space theory, which, in some sense, is the true subject of this book.

Acknowledgement. This book is based on lecture notes for various topics courses given during the last 10 years or so at Texas A&M University. Thanks are due to Robin Campbell, who typed most on them, for her excellent work. I am indebted to Hervé Chapelat, who took notes from my even earlier lectures on H^p -spaces there, for Chapters 3 and 4. The completion of this volume was stimulated by the Winter School on ‘Type, cotype and martingales on Banach spaces and metric spaces’ at IHP (Paris), 2–8 February 2011, for which I would like to thank the organizers. I am very grateful to all those who, at some stage, helped me to correct mistakes and misprints and who suggested improvements of all kinds, in particular Michael Cwikel, Sonia Fourati, Julien Giol, Rostyslav Kravchenko, Bernard Maurey, Adam Osękowski, Javier Parcet, Yanqi Qiu, Mikael de la Salle, Francisco Torres-Ayala, Mateusz Wasilewski, and Quanhua Xu; S. Petermichl for help on Chapter 6; and M. I. Ostrovskii for advice on Chapter 13. I am especially grateful to Mikael de la Salle for drawing all the pictures with TikZ.

Description of the contents

We will now review the contents of this book chapter by chapter.

Chapter 1 begins with preliminary background: we introduce Banach space valued L_p -spaces, conditional expectations and the central notion in this book, namely Banach space valued martingales associated to a filtration $(\mathcal{A}_n)_{n \geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We describe the classical examples of filtrations (the dyadic one and the Haar one) in §1.4. If B is an arbitrary Banach space and the martingale (f_n) is associated to some f in $L_p(B)$ by $f_n = \mathbb{E}^{\mathcal{A}_n}(f)$ ($1 \leq p < \infty$), then, assuming $\mathcal{A} = \mathcal{A}_\infty$ for simplicity, the fundamental convergence theorems say that

$$f_n \rightarrow f$$

both in $L_p(B)$ and almost surely (a.s.).

The convergence in $L_p(B)$ is Theorem 1.14, while the a.s. convergence is Theorem 1.30. The latter is based on Doob's classical *maximal inequalities* (Theorem 1.25), which are proved using the crucial notion of *stopping time*. We also describe the dual form of Doob's inequality due to Burkholder-Davis-Gundy (see Theorem 1.26). Doob's maximal inequality shows that the convergence of f_n to f in $L_p(B)$ 'automatically' implies a.s. convergence. This, of course, is special to martingales, but in general it requires $p \geq 1$. However, for martingales that are sums of independent, symmetric random variables (Y_n) (i.e. we have $f_n = \sum_1^n Y_k$), this result holds for $0 < p < 1$ (see Theorem 1.40). It also holds, roughly, for $p = 0$. This is the content of the celebrated Ito-Nisio theorem (see Theorem 1.43), which asserts that even a weak form of convergence of the series $f_n = \sum_1^n Y_k$ implies its a.s. norm convergence.

In §1.8, we prove, again using martingales, a version of Phillips's theorem. The latter is usually stated as saying that, if B is separable, any countably additive measure on the Borel σ -algebra of B is 'Radon', i.e. the measure of a Borel