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MARTINGALES IN BANACH SPACES

This book is focused on the major applications of martingales to the geometry of Banach spaces, but a substantial discussion of harmonic analysis in Banach space valued Hardy spaces is presented. Exciting links between super-reflexivity and some metric spaces related to computer science are covered, as is an outline of the recently developed theory of non-commutative martingales, which has natural connections with quantum physics and quantum information theory.

Requiring few prerequisites and providing fully detailed proofs for the main results, this self-contained study is accessible to graduate students with basic knowledge of real and complex analysis and functional analysis. Chapters can be read independently, each building from introductory notes, and the diversity of topics included also means this book can serve as the basis for a variety of graduate courses.

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Martingales in Banach Spaces

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Contents

Introduction

Martingales (with discrete time) lie at the centre of this book. They are known to have major applications to virtually every corner of probability theory. Our central theme is their applications to the geometry of Banach spaces.

We should emphasize that we do *not* assume any knowledge about scalar valued martingales. Actually, the beginning of this book gives a self-contained introduction to the basic martingale convergence theorems for which the use of the norm of a vector valued random variable instead of the modulus of a scalar one makes little difference. Only when we consider the 'boundedness implies convergence' phenomenon does it start to matter. Indeed, this requires the Banach space B to have the Radon-Nikodým property (RNP). But even at this point, the reader who wishes to concentrate on the scalar case could simply assume that B is finite-dimensional and disregard all the infinite-dimensional technical points. The structure of the proofs remains pertinent if one does so. In fact, it may be good advice for a beginner to do a first reading in this way. One could argue similarly about the property of 'unconditionality of martingale differences' (UMD): although perhaps the presence of a Banach space norm is more disturbing there, our reader could assume at first reading that B is a Hilbert space, thus getting rid of a number of technicalities to which one can return later.

A major feature of the UMD property is its equivalence to the boundedness of the Hilbert transform (HT). Thus we include a substantial excursion in (Banach space valued) harmonic analysis to explain this.

Actually, connections with harmonic analysis abound in this book, as we include a rather detailed exposition of the boundary behaviour of B-valued harmonic (resp. analytic) functions in connections with the RNP (resp. analytic RNP) of the Banach space B. We introduce the corresponding B-valued Hardy spaces in analogy with their probabilistic counterparts. We are partly motivated by the important role they play in operator theory, when one takes for B the space of bounded operators (or the Schatten p-class) on a Hilbert space.

Hardy spaces are closely linked with martingales via Brownian motion: indeed, for any B-valued bounded harmonic (resp. analytic) function u on the unit disc D, the composition $(u(W_{t\wedge T}))_{t>0}$ of u with Brownian motion stopped before it exits D is an example of a continuous B-valued martingale, and its boundary behaviour depends in general on whether B has the RNP (resp. analytic RNP). We describe this connection with Brownian motion in detail, but we refrain from going too far on that road, remaining faithful to our discrete time emphasis. However, we include short sections summarizing just what is necessary to understand the connections with Brownian martingales in the Banach valued context, together with pointers to the relevant literature. In general, the sections that are a bit far off our main goals are marked by an asterisk. For instance, we describe in §7.1 the Banach space valued version of Fefferman's duality theorem between H^1 and BMO. While this is not really part of martingale theory, the interplay with martingales, both historically and heuristically, is so obvious that we felt we had to include it. The asterisked sections could be kept for a second reading.

In addition to the RN and UMD properties, our third main theme is super-reflexivity and its connections with uniform convexity and smoothness. Roughly, we relate the geometric properties of a Banach space B with the study of the p-variation

$$S_p(f) = \left(\sum\nolimits_1^\infty \|f_n - f_{n-1}\|_B^p\right)^{1/p}$$

of B-valued martingales (f_n). Depending on whether $S_p(f) \in L_p$ is necessary or sufficient for the convergence of (f_n) in $L_p(B)$, we can find an equivalent norm on B with modulus of uniform convexity (resp. smoothness) 'at least as good as' the function $t \to t^p$.

We also consider the strong p-variation

$$V_p(f) = \sup_{0=n(0)<n(1)<n(2)<\cdots} \left(\sum\nolimits_1^\infty \|f_{n(k)} - f_{n(k-1)}\|_B^p\right)^{1/p}$$

of a martingale. For that topic (exceptionally) we devote an entire chapter only to the scalar case. Our crucial tool here is the 'real interpolation method'. Real and complex interpolation in general play an important role in L_p-space theory, so we find it natural to devote a significant amount of space to these two 'methods'.

We allow ourselves several excursions aiming to illustrate the efficiency of martingales, for instance to the concentration of measure phenomenon. We also describe some exciting recent work on non-linear properties of metric spaces analogous to uniform convexity/smoothness and type for metric spaces.

We originally intended to include in this book a detailed presentation of 'non-commutative' martingale theory, but that part became so big that we decided to make it the subject of a (hopefully forthcoming) separate volume to be published, perhaps on the author's web page. We merely outline its contents in the last chapter, devoted to non-commutative L_p-spaces. There the complex interpolation method becomes a central tool.

The book should be accessible to graduate students, requiring only the basics of real and complex analysis (mainly Lebesgue integration) and basic functional analysis (mainly duality, the weak and strong topologies and reflexivity of Banach spaces). Our choice is to give fully detailed proofs for the main results and to indicate references to the refinements in the 'Notes and Remarks' or the asterisked sections. We strive to make the presentations self-contained, and when given a choice, we opt for simplicity over maximal generality. For instance, we restrict the Banach space valued harmonic analysis to functions with domains in the unit disc D or the upper half-plane U in $\mathbb{C}$ (or their boundary $\partial D = \mathbb{T}$ or $\partial U = \mathbb{R}$). We feel the main ideas are easier to grasp in the real or complex uni-dimensional case.

The topics (martingales, H^p-space theory, interpolation, Banach space geometry) are quite diverse and should appeal to several distinct audiences. The main novelty is the choice to bring all these topics together in the various parts of this single volume. We should emphasize that the different parts can be read independently, and each time their start is introductory.

There are natural groupings of chapters, such as 1-2-10-11 or 3-4-5-6 (possibly including parts of 1 and 2, but not necessarily), which could form the basis for a graduate course.

Depending on his or her background, a reader is likely to choose to concentrate on different parts. We hope probabilist graduate students will benefit from the detailed introductory presentation of basic H^p-space theory, its connections with martingales, the links with Banach space geometry and the detailed treatment of interpolation theory (which we illustrate by applications to the strong p-variation of martingales), while graduate students with interest in functional analysis and Banach spaces should benefit more from the initial detailed presentation of basic martingale theory. In addition, we hope to attract readers with interest in computer science wishing to see the sources of the various recent developments on finite metric spaces described in Chapter 13. A reader with an advanced knowledge of harmonic analysis and H^p-theory will probably choose to skip the introductory part on that direction, which is written with non-specialists in mind, and concentrate on the issues specific to Banach space valued functions related to the UMD property and the Hilbert transform.

The choice to include so much background on the real and complex interpolation methods in Chapter 8 is motivated by its crucial importance in Banach space valued L_p-space theory, which, in some sense, is the true subject of this book.

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Description of the contents

We will now review the contents of this book chapter by chapter.

Chapter 1 begins with preliminary background: we introduce Banach space valued L_p-spaces, conditional expectations and the central notion in this book, namely Banach space valued martingales associated to a filtration $(\mathcal{A}_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We describe the classical examples of filtrations (the dyadic one and the Haar one) in §1.4. If B is an arbitrary Banach space and the martingale (f_n) is associated to some f in $L_p(B)$ by $f_n = \mathbb{E}^{\mathcal{A}_n}(f)$ $(1 \leq p < \infty)$, then, assuming $\mathcal{A} = \mathcal{A}_\infty$ for simplicity, the fundamental convergence theorems say that

$$f_n \to f$$

both in $L_p(B)$ and almost surely (a.s.).

The convergence in $L_p(B)$ is Theorem 1.14, while the a.s. convergence is Theorem 1.30. The latter is based on Doob's classical *maximal inequalities* (Theorem 1.25), which are proved using the crucial notion of *stopping time*. We also describe the dual form of Doob's inequality due to Burkholder-Davis-Gundy (see Theorem 1.26). Doob's maximal inequality shows that the convergence of f_n to f in $L_p(B)$ 'automatically' implies a.s. convergence. This, of course, is special to martingales, but in general it requires $p \geq 1$. However, for martingales that are sums of independent, symmetric random variables (Y_n) (i.e. we have $f_n = \sum_1^n Y_k$), this result holds for $0 < p < 1$ (see Theorem 1.40). It also holds, roughly, for $p = 0$. This is the content of the celebrated Ito-Nisio theorem (see Theorem 1.43), which asserts that even a weak form of convergence of the series $f_n = \sum_1^n Y_k$ implies its a.s. norm convergence.

In §1.8, we prove, again using martingales, a version of Phillips's theorem. The latter is usually stated as saying that, if B is separable, any countably additive measure on the Borel σ-algebra of B is 'Radon', i.e. the measure of a Borel subset can be approximated by that of its compact subsets. In §1.9, we prove the strong law of large numbers using the a.s. convergence of reverse B-valued martingales. In §1.10, we give a brief introduction to continuous-time martingales. We mainly explain the basic approximation technique by which one passes from discrete to continuous parameter.

To get to a.s. convergence, all the preceding results need to assume in the first place some form of convergence, e.g. in $L_p(B)$. In classical (i.e. real valued) martingale theory, it suffices to assume *boundedness* of the martingale $\{f_n\}$ in L_p $(p \geq 1)$ to obtain its a.s. convergence (as well as norm convergence, if $1 < p < \infty$). However, this 'boundedness $\Rightarrow$ convergence' phenomenon no longer holds in the B-valued case unless B has a specific property called the Radon-Nikodým property (RNP in short), which we introduce and study in Chapter 2. The RNP of a Banach space B expresses the validity of a certain form of the Radon-Nikodým theorem for B-valued measures, but it turns out to be equivalent to the assertion that all martingales bounded in $L_p(B)$ converge a.s. (and in $L_p(B)$ if $p > 1$) for some (or equivalently all) $1 \leq p < \infty$. Moreover, the RNP is equivalent to a certain 'geometric' property called 'dentability'. All this is included in Theorem 2.9. The basic examples of Banach spaces with the RNP are the reflexive ones and separable duals (see Corollary 2.15).

Moreover, a dual space B^* has the RNP iff the classical duality $L_p(B)^* = L_{p'}(B^*)$ is valid for some (or all) $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (see Theorem 2.22). Actually, for a general B, one can also describe $L_p(B)^*$ as a space of martingales bounded in $L_{p'}(B^*)$, but in general, the latter is larger than the (Bochner sense) space $L_{p'}(B^*)$ itself (see Proposition 2.20). In many situations, it is preferable to have a description of $L_p(B)^*$ as a space of B^*-valued measures or functions (rather than martingales). For that purpose, we give in §2.4 two alternate descriptions of the latter space, either as a space of B^*-valued vector measures, denoted by $\Lambda_{p'}(B^*)$, or, assuming B separable, as a space of weak* measurable B^*-valued functions denoted by $\underline{\Lambda}_{p'}(B^*)$.

In §2.5, we discuss the Krein-Milman property (KMP): this says that any bounded closed convex set $C \subset B$ is the closed convex hull of its extreme points. This is closely related to dentability, but although it is known that RNP $\Rightarrow$ KMP (see Theorem 2.34), the converse implication is still open.

In §2.6, we present a version of Choquet's representation theorem of the points of C as barycenters of measures supported by the extreme points. Choquet's classical result applies only to convex compact sets, while this version requires that the closed bounded separable convex set C lies in an RNP space. The proof is based on martingale convergence.

In §2.7, we prove that, if B is separable, a bounded B-valued vector measure admits a RN density iff the associated linear operator $L_1 \to B$ factorizes through the space ℓ_1 of absolutely summable scalar sequences. This fact (due to Lewis and Stegall) is remarkable because ℓ_1 is the prototypical example of a separable dual (a fortiori ℓ_1 has the RNP). In other words, if B has the RNP, the bounded B-valued vector measures 'come from' ℓ_1 valued vector measures, and the latter are differentiable since ℓ_1 itself has the RNP. There is also a version of this result when B is not separable (see Theorem 2.39).

In Chapter 3, we introduce the Hardy space $h^p(D; B)$ $(1 \le p \le \infty)$ formed of all the B-valued harmonic functions $u\colon D \to B$ (on the unit disc $D \subset \mathbb{C}$) such that

$$\|u\|_{h^p(D;B)} = \sup_{0<r<1} \left(\int \|u(re^{it})\|^p dm(t) \right)^{1/p} < \infty.$$

When $p = \infty$, this is just the space of bounded harmonic functions $u\colon D \to B$.

In analogy with the martingale case treated in Chapter 1, to any f in $L_p(\mathbb{T}; B)$, we can associate a harmonic function $u\colon D \to B$ that extends f in the sense that $u(re^{it}) \to f(e^{it})$ for almost all e^{it} in $\partial D = \mathbb{T}$, and also (if $p < \infty$)

$$\int \|u(re^{it}) - f(e^{it})\|^p dm(t) \to 0$$

when $r \uparrow 1$ (see Theorem 3.1). The convergence at almost all boundary points requires a specific radial maximal inequality, which we derive from the classical Hardy-Littlewood maximal inequality. Actually, we present this in the framework of 'non-tangential' maximal inequalities. The term 'non-tangential' refers to the fact that we study the limit of $u(z)$ (in the norm of B) when z tends to e^{it} but staying inside a cone with vertex e^{it} and opening angle $\beta < \frac{\pi}{2}$.

This topic is closely linked with Brownian motion (see especially [176]). Indeed, the paths of a complex valued Brownian motion $(W_t)_{t>0}$ (starting at the origin) almost surely cross the boundary of the unit disc D in finite time, and if we define $T_r = \inf\{t > 0 \,\big|\, |W_t| = r\}$, then for any u in $h^p(D; B)$, the random variables $\{u(W_{T_r}) \mid 0 < r < 1\}$ form a martingale bounded in $L_p(B)$, for which the maximal function is closely related to the non-tangential one of u.

In general, the latter B-valued martingales do not converge. However, we show in Corollary 3.31 that they do so if B has the RNP. Actually, the RNP of B is equivalent to the a.e. existence of radial (or non-tangential) limits or of limits along almost all Brownian paths, for the functions in $h^p(D; B)$ $(1 \le p \le \infty)$ (see Theorem 3.25 and Corollary 3.31 for this). In the supplementary §4.7, we compare the various modes of convergence to the boundary, radial, non-tangential or Brownian for a general B-valued harmonic function. This brief §4.7 outlines some beautiful work by Burkholder, Gundy and Silverstein [176] and Brossard [148].

In Chapter 4 we turn to the subspace $H^p(D; B) \subset h^p(D; B)$ formed of all the *analytic* B-valued functions in $h^p(D; B)$. One major difference is that the (radial or non-tangential) maximal inequalities now extend to all values of p including $0 < p \leq 1$. To prove this, we make crucial use of *outer functions*; this gives us a convenient factorization of any f in $H^p(D; B)$ as a product $f = Fg$ with a *scalar* valued F in H^p and g in $H^\infty(D; B)$ (see Theorem 4.15). The Hardy spaces $H^p(D; B)$ naturally lead us to a more general form of RNP called the analytic RNP (ARNP in short). This is equivalent to the a.e. existence of radial (or non-tangential) limits for all functions in $H^p(D; B)$ for some (or all) $0 < p \leq \infty$. Thus we have RNP $\Rightarrow$ ARNP, but the converse fails, for instance (see Theorem 4.32) $B = L_1([0, 1])$ has the ARNP but not the RNP. On the martingale side, the strict inclusion

$$\{\text{analytic}\} \subsetneqq \{\text{harmonic}\}$$

admits an analogue involving the notion of 'analytic martingale' or 'Hardy martingale', and the convergence of the latter (with the usual bounds) is equivalent to the ARNP (see Theorem 4.30). In §4.6 we briefly review the analogous Banach space valued H^p-space theory for functions on the upper half-plane

$$U = \{z \in \mathbb{C} | \Im(z) > 0\}.$$

Chapter 5 is devoted to the UMD property. After a brief presentation of Burkholder's inequalities in the scalar case, we concentrate on their analogue for Banach space valued martingales (f_n). In the scalar case, when $1 < p < \infty$, we have

$$\sup_n \|f_n\|_p \simeq \|\sup |f_n|\|_p \simeq \|S(f)\|_p,$$

where $S(f) = (|f_0|^2 + \sum |f_n - f_{n-1}|^2)^{1/2}$, and where $A_p \simeq B_p$ means that there are positive constants C'_p and C''_p such that $C'_p A_p \leq B_p \leq C''_p A_p$. In the Banach space valued case, we replace $S(f)$ by

$$R(f)(\omega) = \sup\nolimits_N \left(\int \left\| f_0(\omega) + \sum\nolimits_1^N \varepsilon_n (f_n - f_{n-1})(\omega) \right\|^2 d\nu \right)^{1/2}, \tag{1}$$

where ν is the uniform probability measure on the set Δ of all choices of signs $(\varepsilon_n)_n$ with $\varepsilon_n = \pm 1$.

In §5.2 we prove Kahane's inequality, i.e. the equivalence of all the L_p-norms for series of the form $\sum_1^\infty \varepsilon_n x_n$ with x_n in an arbitrary Banach space when $0 < p < \infty$ (see (5.16)); in particular, up to equivalence, we can substitute to the L_2-norm in (1) any other L_p-norm for $p < \infty$.

Let $\{x_n\}$ be a sequence in a Banach space such that the series $\sum \varepsilon_n x_n$ converges almost surely. We set

$$R(\{x_n\}) = \left(\int_D \left\| \sum \varepsilon_n x_n \right\|^2 d\nu \right)^{1/2}.$$

With this notation we have

$$R(f)(\omega) = R(\{f_0(\omega), f_1(\omega) - f_0(\omega), f_2(\omega) - f_1(\omega), \cdots\}). \tag{2}$$

The UMD_p and UMD properties are introduced in §5.3. Consider the series

$$\tilde{f}_\varepsilon = f_0 + \sum\nolimits_1^\infty \varepsilon_n (f_n - f_{n-1}). \tag{3}$$

By definition, when B is UMD_p, (f_n) converges in $L_p(B)$ iff (3) converges in $L_p(B)$ for all choices of signs $\varepsilon_n = \pm 1$ or equivalently iff it converges for almost all (ε_n). Moreover, we have then for $1 < p < \infty$ and all choices of signs $\varepsilon = (\varepsilon_n)$

$$\|\tilde{f}_\varepsilon\|_{L_p(B)} \simeq \|f\|_{L_p(B)} \tag*{$(4)_p$}$$

$$\sup_{n \geq 0} \|f_n\|_{L_p(B)} \simeq \|R(f)\|_p. \tag*{$(5)_p$}$$

See Proposition 5.10. The case $p = 1$ (due to Burgess Davis) is treated in §5.6. The main result of §5.3 is the equivalence of UMD_p and UMD_q for any $1 < p, q < \infty$. We give two proofs of this; the first one is based on distributional (also called 'good λ') inequalities. This is an extrapolation principle, which allows us to show that, for a given Banach space B, $(4)_q \Rightarrow (4)_p$ for any $1 < p < q$. In the scalar case one starts from the case $q = 2$, which is obvious by orthogonality, and uses the preceding implication to deduce from it the case $1 < p < 2$ and then $2 < p < \infty$ by duality. We also give a more delicate variant of the extrapolation principle that avoids duality and deduces the desired inequality for any $1 < p < \infty$ from a certain form of weak-type estimate involving pairs of stopping times (see Lemma 5.26).

The second proof is based on Gundy's decomposition, which is a martingale version of the Calderón-Zygmund decomposition in classical harmonic analysis. There one proves a weak-type (1,1) estimate and then invokes the Marcinkiewicz theorem to obtain the case $1 < p < 2$. We describe the latter in an appendix to Chapter 5.

In §5.8 we show that to check that a space B is UMD_p, we may restrict ourselves to martingales adapted to the *dyadic* filtration, and the associated UMD-constant remains the same. The proof is based on a result of independent interest: if $p < \infty$, any finite martingale $(f_0, \ldots, f_N)$ in $L_p(B)$ (on a large enough probability space) can be approximated in $L_p(B)$ by a subsequence of a dyadic martingale.

In §5.9 we prove the Burkholder-Rosenthal inequalities. In the scalar case, this boils down to the equivalence

$$\sup_n \|f_n\|_p \simeq \|\sigma(f)\|_p + \| \sup_n |f_n - f_{n-1}|\|_p,$$

where $\sigma(f) = (|f_0|^2 + \sum \mathbb{E}_{n-1}|f_n - f_{n-1}|^2)^{1/2}$, valid for $2 \le p < \infty$.

Rosenthal originally proved this when f_n is a sum of independent variables, and Burkholder extended it to martingales. We describe a remarkable example of complemented subspace of L_p (the Rosenthal space X_p), which motivated Rosenthal's work.

In §5.10 we describe Stein's inequality and its B-valued analogue when B is a UMD Banach space. Let $(\mathcal{A}_n)_{n\ge 0}$ be a filtration as usual, and let $(x_n)_{n\ge 0}$ be now an *arbitrary* sequence in L_p. Let $y_n = \mathbb{E}^{\mathcal{A}_n} x_n$. Stein's inequality asserts that for any $1 < p < \infty$, there is a constant C_p such that

$$\left\| \left(\sum |y_n|^2 \right)^{1/2} \right\|_p \le C_p \left\| \left(\sum |x_n|^2 \right)^{1/2} \right\|_p \tag{6}$$

for any (x_n) in L_p.

For x_n in $L_p(B)$, with B UMD, the same result remains valid if we replace on both sides of (6) the expression $(\sum |x_n|^2)^{1/2}$ by

$$\left(\int \left\| \sum \varepsilon_n x_n \right\|_B^2 d\nu \right)^{1/2}.$$

See (5.89).

In §5.11 we describe Burkholder's geometric characterization of UMD spaces in terms of ζ-convexity (Theorem 5.64). We also include a more recent result (Theorem 5.69) in the same vein. The latter asserts that a real Banach space of the form $B = X \oplus X^*$ is UMD iff the function

$$x \oplus \xi \to \xi(x)$$

is the difference of two real valued convex continuous functions on B.

We end Chapter 5 by a series of appendices. In §5.12 we prove the hypercontractive inequalities on $\{-1, 1\}$. In §5.13 we discuss the Hölder-Minkowski inequality, which says that we have a norm 1 inclusion

$$L_q(m'; L_p(m)) \subset L_p(m; L_q(m'))$$

when $p \ge q$. In §5.14 we give some basic background on the space weak-L_p, usually denoted $L_{p,\infty}$, and §5.16 is devoted to a quick direct proof of the Marcinkiewicz theorem. In §5.15 we present a trick frequently used by Burkholder that we call the reverse Hölder principle. Hölder's inequality for a random variable Z on a probability space tells us that $Z \in L_q \Rightarrow Z \in L_r$ when $r < q$. The typical reverse Hölder principle shows that a suitable L_r bound involving independent copies of Z implies conversely that Z is in weak-L_p (or $L_{p,\infty}$) and a fortiori in L_q for all $r < q < p$.

In the final appendix to this chapter, §5.17, we explain why certain forms of exponential integrability of a function f can be equivalently reformulated in terms of the growth of the L_p-norms of f. This shows that the growth when $p \to \infty$ of the constants in many of the martingale inequalities we consider often implies an exponential inequality.

In Chapter 6 we start with some background on the Hilbert transform. We say that a Banach space B is an HT-space if the Hilbert transform on $\mathbb{T}$ defines a bounded operator on $L_p(\mathbb{T}; B)$ for some (or equivalently all) $1 < p < \infty$. We then show that UMD and HT are equivalent properties. To show HT $\Rightarrow$ UMD, we follow Bourgain's well-known argument (see §6.2). The converse implication UMD $\Rightarrow$ HT was originally proved using ideas derived from the beautiful observation that the Hilbert transform can be viewed as a sort of martingale transform relative to stochastic integrals over Brownian motion. We merely outline this proof in §6.4, and instead, we present first, in full details the more recent remarkable proof from Petermichl [378], which uses only martingale transforms relative to Haar systems, but with respect to a randomly chosen dyadic filtration (see §6.3).

In §6.5, following Bourgain, we prove that the Littlewood-Paley inequalities are valid in the B-valued case if B is UMD. More precisely, consider the formal Fourier series

$$f = \sum\nolimits_{n \in \mathbb{Z}} \hat{f}(n) e^{int}$$

of a function in $L_p(B)$, and let

$$V(f) = (\|\hat{f}(0)\|_B^2 + R(\{\Delta_n^+\})^2 + R(\{\Delta_n^-\})^2)^{1/2},$$

where

$$\Delta_n^+ = \sum\nolimits_{2^n \le k < 2^{n+1}} \hat{f}(k) e^{int} \quad \text{and} \quad \Delta_n^- = \sum\nolimits_{2^n \le -k < 2^{n+1}} \hat{f}(k) e^{ikt}.$$

The B-valued version of the Littlewood-Paley inequality is the equivalence of $\|f\|_{L_p(B)}$ and $\|V(f)\|_p$.

In Theorem 6.35 we show that the analogue of the Hilbert transform for the compact group $\{-1, 1\}^{\mathbb{N}}$ (in place of $\mathbb{T}$) is bounded on $L_p(B)$ for any $1 < p < \infty$ iff B is UMD. This is the analogue of the implication UMD $\Rightarrow$ HT, but for the Walsh system in place of the trigonometric one. The proof of this case is much more transparent.

In §6.7, we briefly review the analytic UMD property (AUMD). This is a weakening of UMD obtained by restricting to either analytic or Hardy martingales. The latter are discretizations of the B-valued martingales obtained by composing complex Brownian motion with a B-valued analytic function on D. The main novelty is that the space $B = L_1$ itself has the AUMD property. Moreover, the latter implies the ARNP.

In §7.1 we describe the B-valued version of Fefferman's famous duality theorem between H^1 and BMO on $\mathbb{T}$ (resp. $\mathbb{R}$). This requires that we carefully identify the various B-valued analogues of the Hardy spaces H^1 or BMO. If B is arbitrary, the duality holds provided that we use the atomic version of B-valued H^1, denoted by $h^1_{\text{at}}(\mathbb{T}; B)$ (resp. $h^1_{\text{at}}(\mathbb{R}; B)$); this is the so-called real variable variant of H^1, as opposed to the more classical Hardy space of analytic function, denoted by $H^1(D; B)$ (resp. $H^1(U; B)$) on the unit disc D (resp. the upper half-plane U). In general, for any $f \in H^1(D; B)$ (resp. $H^1(U; B)$), the boundary values of f and its Hilbert transform $\tilde{f}$ are both in $h^1_{\text{at}}(\mathbb{T}; B)$ (resp. $h^1_{\text{at}}(\mathbb{R}; B)$). When B is UMD (and only then), the converse also holds (see Corollary 7.20).

In §7.3 we discuss the space BMO and the B-valued version of H^1 in the martingale context. This leads naturally to the atomic version of B-valued H^1, denoted by $h^1_{at}(\{\mathcal{A}_n\}; B)$ with respect to a filtration $\{\mathcal{A}_n\}$. Its dual can be identified with a BMO-space for B^*-valued martingales, at least for a 'regular' filtration $(\mathcal{A}_n)$. Equivalently (see Theorem 7.32), the space $h^1_{at}(B)$ can be identified with $\widetilde{h}^1_{\max}(B)$, which is defined as the completion of $L_1(B)$ with respect to the norm $f \mapsto \mathbb{E} \sup_n \|f_n\|_B$ (here $f_n = \mathbb{E}^{\mathcal{A}_n} f$).

In §7.5 we show that the classical BMO space coincides with the intersection of two suitably chosen translates of the dyadic martingale-BMO space.

In Chapter 8 we describe successively the complex and the real method of interpolation for pairs of Banach spaces (B_0, B_1) assumed compatible for interpolation purposes. The complex interpolation space is denoted by $(B_0, B_1)_\theta$. It depends on the single parameter $0 < \theta < 1$ and requires B_0, B_1 to be both *complex* Banach spaces. Complex interpolation is a sort of 'abstract' generalization of the classical Riesz-Thorin theorem, asserting that if an operator T has norm 1 simultaneously on both spaces $B_0 = L_{p_0}$ and $B_1 = L_{p_1}$, with $1 \le p_0 < p_1 \le \infty$, then it also has norm 1 on the space L_p for any p such that $p_0 < p < p_1$. Since we make heavy use of this *complex method* in noncommutative L_p-theory, we review its basic properties somewhat extensively.

The real interpolation space is denoted by $(B_0, B_1)_{\theta,q}$. It depends on two parameters $0 < \theta < 1$, $1 \le q \le \infty$, and now (B_0, B_1) can be a pair of *real* Banach spaces. Real interpolation is a sort of abstract generalization of the

Marcinkiewicz classical theorem already proved in an appendix to Chapter 5. The real interpolation space is introduced using the 'K-functional' defined, for any $B_0 + B_1$, by

$$\forall t > 0 \;\; K_t(x) = \inf\{\|x_0\|_{B_0} + t\|x_1\|_{B_1} \mid x_0 \in B_0, x_1 \in B_1, x = x_0 + x_1\}.$$

When $B_0 = L_1(\Omega, m)$, $B_1 = L_\infty(\Omega, m)$, we find

$$K_t(x) = \int_0^t x^\dagger(s)ds, \tag{7}$$

where $x^\dagger$ is the non-increasing rearrangement of $|x|$ and (Ω, m) is an arbitrary measure space. We prove this in Theorem 8.50 together with the identification of $(L_1, L_\infty)_{\theta,q}$ with the Lorentz space $L_{p,q}$ for $p = (1-\theta)^{-1}$.

Real interpolation will be crucially used in the later Chapters 9 and 12 in connection with our study of the 'strong p-variation' of martingales. The two interpolation methods satisfy distinct properties but are somewhat parallel to each other. For instance, duality, reiteration and interpolation between vector valued L_p-spaces are given parallel treatments in Chapter 8.

The classical reference on interpolation is [6] (see also [51]). Given the existence of these excellent treatises, one could question the need for the compact but detailed presentation (essentially self-complete) we give in Chapter 8. We hope this will be appreciated at least by some readers (e.g. interpolation theory is not the usual background of probabilists). Moreover, several specific points which play an important role for Banach space valued L_p-spaces (commutative or not) are not completely proved in the existing references (which contain much more on other topics). For instance, we have in mind (8.15), Theorem 8.34, Theorem 8.39, §8.6, §8.7 and §8.11.

We devote a significant amount of space in §8.3 to describing the dual of the complex interpolation space $(B_0, B_1)_\theta$ in connection with the RNP and ARNP. The analogous question for the real method, involving the J-functional that is dual to $x \mapsto K_t(x)$, is treated in §8.8.

We devote much space to pairs of Banach space valued L_p-spaces; i.e. given a compatible pair (B_0, B_1), a measure space $(\Omega, \mathcal{A}, m)$ and $1 \le p_0, p_1 \le \infty$, we identify the interpolation spaces for the pair $(Lp_0(m; B_0), Lp_1(m; B_1))$. The complex case is treated in Theorem 8.21. For the real case, we give a formula for the K-functional (see §8.6), generalizing (7), from which the interpolation space can be derived easily (see §8.7) in connection with Lorentz spaces.

Lastly, we consider pairs with some special symmetries, e.g. pairs of the form (B^*, B) associated to a positive definite inclusion $B^* \subset B$ (or $\overline{B^*} \subset B$ in the complex case) for which we show that Hilbert space automatically appears at the centre of the interpolation scale.

In Chapter 9 we study the strong p-variation $W_p(f)$ of a scalar martingale (f_n). This is defined as the supremum of

$$\left(|f_{n(0)}|^p + \sum_{k=1}^{\infty} |f_{n(k)} - f_{n(k-1)}|^p\right)^{1/p}$$

over all possible increasing sequences

$$0 = n(0) < n(1) < n(2) < \cdots .$$

The main results are Theorem 9.2 and Proposition 9.6. Roughly, this says that, if $1 \le p < 2$, $W_p(f)$ is essentially 'controlled' by $(\sum |f_n - f_{n-1}|^p)^{1/p}$, i.e. by the finest partition corresponding to consecutive $n(k)$s, while, in sharp contrast, if $2 < p < \infty$, it is 'controlled' by $|f_\infty| = \lim |f_n|$, or equivalently, by the coarsest partition corresponding to the choice $n(0) = 0$, $n(1) = \infty$.

The proofs combine a simple stopping time argument with the reiteration theorem of the real interpolation method.

In Chapter 10, we turn to uniform convexity and uniform smoothness of Banach spaces. We show that certain martingale inequalities characterize the Banach spaces B that admit an equivalent norm for which there is a constant C and $2 \le q < \infty$ or $1 < p \le 2$ such that, for any x, y in B,

$$\|x\|^q + C\|y\|^q \le \frac{\|x+y\|^q + \|x-y\|^q}{2} \tag{8}$$

or

$$\frac{\|x+y\|^p + \|x-y\|^p}{2} \le \|x\|^p + C\|y\|^p, \tag{9}$$

respectively. This is the content of Corollary 10.7 (resp. Corollary 10.23). We use this in Theorem 10.1 (resp. Theorem 10.25) to show that actually *any* uniformly convex (resp. smooth) Banach space admits for some $2 \le q < \infty$ (resp. $1 < p \le 2$) such an equivalent renorming. The inequality (8) (resp. (9)) holds iff the modulus of uniform convexity (resp. smoothness) $\delta(\varepsilon)$ (resp. $\rho(t)$) satisfies $\inf_{\varepsilon>0} \delta(\varepsilon)\varepsilon^{-q} > 0$ (resp. $\sup_{t>0} \rho(t)t^{-p} < \infty$). In that case we say that the space is q-uniformly convex (resp. p-uniformly smooth). The proof also uses inequalities going back to Gurarii, James and Lindenstrauss on monotone basic sequences. We apply the latter to martingale difference sequences viewed as monotone basic sequences in $L_p(B)$. Our treatment of uniform smoothness in §10.2 runs parallel to that of uniform convexity in §10.1.

In §10.3 we estimate the moduli of uniform convexity and smoothness of L_p for $1 < p < \infty$. In particular, L_p is p-uniformly convex if $2 \le p < \infty$ and p-uniformly smooth if $1 < p \le 2$.

In §10.5 we prove analogues of Burkholder's inequalities, but with the square function now replaced by

$$S_p(f) = \left(\|f_0\|_B^p + \sum\nolimits_1^\infty \|f_n - f_{n-1}\|_B^p\right)^{1/p}.$$

Unfortunately, the results are now only one sided: if B satisfies (8) (resp. (9)), then $\|S_q(f)\|_r$ is dominated by (resp. $\|S_p(f)\|_r$ dominates) $\|f\|_{L_r(B)}$ for all $1 < r < \infty$, but here $p \le 2 \le q$ and the case $p = q$ is reduced to the Hilbert space case.

In §10.6 we return to the strong p-variation and prove analogous results to the preceding ones, but this time with $W_q(f)$ and $W_p(f)$ in place of $S_q(f)$ and $S_p(f)$ and $1 < p < 2 < q < \infty$. The technique here is similar to that used for the scalar case in Chapter 9.

Chapter 11 is devoted to *super-reflexivity*. A Banach space B is super-reflexive if every space that is *finitely representable* in B is reflexive. In §11.1 we introduce finite representability and general super-properties in connection with ultraproducts. We include some background about the latter in an appendix to Chapter 11.

In §11.2 we concentrate on super-P when P is either 'reflexivity' or the RNP. We prove that super-reflexivity is equivalent to the super-RNP (see Theorem 11.11). We give (see Theorem 11.10) a fundamental characterization of reflexivity, from which one can also derive easily (see Theorem 11.22) one of super-reflexivity.

As in the preceding chapter, we replace B by $L_2(B)$ and view martingale difference sequences as monotone basic sequences in $L_2(B)$. Then we deduce the martingale inequalities from those satisfied by general basic sequences in super-reflexive spaces. For that purpose, we review a number of results about basic sequences that are not always directly related to our approach. For instance, we prove the classical fact that a Banach space with a basis is reflexive iff the basis is both boundedly complete and shrinking. While we do not directly use this, it should help the reader understand why super-reflexivity implies inequalities of the form either $(\sum \|x_n\|^q)^{1/q} \le C\|\sum x_n\|$ for $q < \infty$ or $\|\sum x_n\| \le (\sum \|x_n\|^p)^{1/p}$ for $p > 1$ (see (iv) and (v) in Theorem 11.22). Indeed, they can be interpreted as a strong form of 'boundedly complete' for the first one and of 'shrinking' for the second one.

In §11.3 we show that uniformly non-square Banach spaces are reflexive, and hence automatically super-reflexive (see Theorem 11.24 and Corollary 11.26). More generally, we go on to prove that B is super-reflexive iff it is J-convex, or equivalently iff it is J-(n, ε) convex for some $n \ge 2$ and some $\varepsilon > 0$. We say that B is J-(n, ε) convex if, for any n-tuple $(x_1, \dots, x_n)$ in the unit ball of B, there is an integer $j = 1, \ldots, n$ such that

$$\left\| \sum_{i<j} x_i - \sum_{i\geq j} x_i \right\| \leq n(1-\varepsilon).$$

When $n = 2$, we recover the notion of 'uniformly non-square'. The implication super-reflexive $\Rightarrow$ J-convex is rather easy to derive (as we do in Corollary 11.34) from the fundamental reflexivity criterion stated as Theorem 11.10. The converse implication (due to James) is much more delicate. We prove it following essentially the Brunel-Sucheston approach ([122]), which in our opinion is much easier to grasp. This construction shows that a non-super-reflexive (or merely non-reflexive) space B contains very *extreme finite-dimensional* structures that constitute obstructions to either reflexivity or the RNP. For instance, any such B admits a space $\widetilde{B}$ finitely representable in B for which there is a dyadic martingale (f_n) with values in the unit ball of $\widetilde{B}$ such that

$$\forall n \geq 1 \qquad \|f_n - f_{n-1}\|_B \equiv 1.$$

Thus the unit ball of $\widetilde{B}$ contains an extremely sparsely separated infinite dyadic tree. (See Remark 1.35 for concrete examples of such trees.)

In §11.4 we finally connect super-reflexivity and uniform convexity. We prove that B is super-reflexive iff it is isomorphic to either a uniformly convex space, or a uniformly smooth one, or a uniformly non-square one. By the preceding Chapter 10, we already know that the renormings can be achieved with moduli of convexity and smoothness of 'power type'. Using interpolation (see Proposition 11.44), we can even obtain a renorming that is both p-uniformly smooth and q-uniformly convex for some $1 < p, q < \infty$, but it is still open whether this holds with the optimal choice of $p > 1$ and $q < \infty$. To end Chapter 11, we give a characterization of super-reflexivity by the validity of a version of the strong law of large numbers for B-valued martingales.

In §11.6 we discuss the stability of super-reflexivity (as well as uniform convexity and smoothness) by interpolation. We also show that a Banach lattice is super-reflexive iff it is isomorphic, for some $\theta > 0$, to an interpolated space $(B_0, B_1)_\theta$ where B_1 is a Hilbert space (and B_0 is arbitrary). Such spaces are called θ-Hilbertian.

In §11.7 we discuss the various complex analogues of uniform convexity: When restricted to analytic (or Hardy, or PSH) martingales, the martingale inequalities characterizing p-uniform convexity lead to several variants of uniform convexity, where roughly convex functions are replaced by plurisubharmonic ones. Of course, this subject is connected to the analytic RNP and analytic UMD, but many questions remain unanswered.

In Chapter 12 we study the real interpolation spaces $(v_1, \ell_\infty)_{\theta,q}$ between the space v_1 of sequences with bounded variation and the space ℓ_∞ of bounded sequences. Explicitly, v_1 (resp. ℓ_∞) is the space of scalar sequences (x_n) such that $\sum_1^\infty |x_n - x_{n-1}| < \infty$ (resp. $\sup |x_n| < \infty$) equipped with its natural norm. The inclusion $v_1 \to \ell_\infty$ plays a major part (perhaps behind the scene) in our treatment of (super-) reflexivity in Chapter 11. Indeed, by the fundamental Theorem 11.10, B is non-reflexive iff the inclusion $\mathcal{J}: v_1 \to \ell_\infty$ factors through B, i.e. it admits a factorization

$$v_1 \xrightarrow{a} B \xrightarrow{b} \ell_\infty,$$

with bounded linear maps a, b such that $\mathcal{J} = ba$.

The remarkable work of James on J-convexity (described in Chapter 11) left open an important point: whether any Banach space B such that ℓ_1^n is not finitely representable in B (i.e. is not almost isometrically embeddable in B) must be reflexive. James proved that the answer is yes if $n = 2$, but for $n > 2$, this remained open until James himself settled it [281] by a counter-example for $n = 3$ (see also [283] for simplifications). In the theory of type (and cotype), it is the same to say that, for some $n \geq 2$, B does not contain ℓ_1^n almost isometrically or to say that B has type p for some $p > 1$ (see the survey [347]). Moreover, type p can be equivalently defined by an inequality analogous to that of p-uniform smoothness, but only for martingales with independent increments. Thus it is natural to wonder whether the strongest notion of 'type p', namely type 2, implies reflexivity. In another tour de force, James [282] proved that *it is not so*. His example is rather complicated. However, it turns out that the real interpolation spaces $\mathcal{W}_{p,q} = (v_1, \ell_\infty)_{\theta,q}$ $(1 < p, q < \infty, 1 - \theta = 1/p)$ provide very nice examples of the same kind. Thus, following [399], we prove in Corollary 12.19 that $\mathcal{W}_{p,q}$ has exactly the same type and cotype exponents as the Lorentz space $\ell_{p,q} = (\ell_1, \ell_\infty)_{\theta,q}$ as long as $p \neq 2$, although $\mathcal{W}_{p,q}$ is *not reflexive* since it lies between v_1 and ℓ_∞. The singularity at $p = 2$ is necessary since (unlike $\ell_2 = \ell_{2,2}$) the space $\mathcal{W}_{2,2}$, being non-reflexive, cannot have both type 2 and cotype 2 since that would force it to be isomorphic to Hilbert space.

A key idea is to consider similarly the B-valued spaces

$$\mathcal{W}_{p,q}(B) = (v_1(B), \ell_\infty(B))_{\theta,q},$$

where B can be an arbitrary Banach space. When $p = q$, we set

$$\mathcal{W}_p(B) = \mathcal{W}_{p,p}(B).$$

We can derive the type and cotype of $\mathcal{W}_{p,q}$ in two ways. The first one proves that the vector valued spaces $\mathcal{W}_{p,q}(L_r)$ satisfy the same kind of 'Hölder-Minkowski' inequality as the Lorentz spaces $\ell_{p,q}$, with the only exception of

$p = r$. This is the substance of Corollary 12.18 (and Corollary 12.27): we have the inclusion

$$\mathcal{W}_p(L_r) \subset L_r(\mathcal{W}_p) \quad \text{if} \quad r > p$$

and the reverse inclusion if $p < r$.

Another way to prove this (see Remark 12.28) goes through estimates of the K-functional for the pairs (v_1, ℓ_∞) and (v_r, ℓ_∞) for $1 < r < \infty$ (see Lemma 12.24). Indeed, by the reiteration theorem, we may identify $(v_1, \ell_\infty)_{\theta,q}$ and $(v_r, \ell_\infty)_{\theta',q}$ if $1 - \theta = (1 - \theta')/r$, and similarly in the Banach space valued case (see Theorem 12.25), we have

$$(v_1(B), \ell_\infty(B))_{\theta,q} = (v_r(B), \ell_\infty(B))_{\theta',q}.$$

We also use reiteration in Theorem 12.14 to describe the space $(v_r, \ell_\infty)_{\theta,q}$ for $0 < r < 1$. In the final Theorem 12.31, we give an alternate description of $\mathcal{W}_p = \mathcal{W}_{p,p}$, which should convince the reader that it is a very natural space (this is closely connected to 'splines' in approximation theory). The description is as follows: a sequence $x = (x_n)_n$ belongs to $\mathcal{W}_p$ iff $\sum_N S_N(x)^p < \infty$, where $S_N(x)$ is the distance in ℓ_∞ of x from the subspace of all sequences (y_n) such that $\text{card}\{n \mid |y_n - y_{n-1}| \neq 0\} \leq N$.

In Chapter 12 we also include a discussion of the classical James space (usually denoted by J), which we denote by v_2^0. The spaces $\mathcal{W}_{p,q}$ are in many ways similar to the James space; in particular, if $1 < p, q < \infty$, they are of co-dimension 1 in their bidual (see Remark 12.8).

In §13.1 and §13.2 we present applications of a certain exponential inequality (due to Azuma) to concentration of measure for the symmetric groups and for the Hamming cube.

In Chapter 13 we give two characterizations of super-reflexive Banach spaces by properties of the underlying *metric spaces*. The relevant properties involve finite metric spaces. Given a sequence $\mathcal{T} = (T_n, d_n)$ of finite metric spaces, we say that the sequence $\mathcal{T}$ embeds Lipschitz uniformly in a metric space (T, d) if, for some constant C, there are subsets $\widetilde{T}_n \subset T$ and bijective mappings $f_n : T_n \to \widetilde{T}_n$ with Lipschitz norms satisfying

$$\sup_n \|f_n\|_{\text{Lip}} \|f_n^{-1}\|_{\text{Lip}} < \infty.$$

Consider for instance the case when T_n is a finite dyadic tree restricted to its first $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ points viewed as a graph and equipped with the usual geodesic distance. In Theorem 13.10 we prove following [138] that a Banach space B is super-reflexive iff it does not contain the sequence of these dyadic trees Lipschitz uniformly. More recently (cf. [289]), it was proved that the trees can be replaced in this result by the '*diamond graphs*'. We describe the analogous characterization with diamond graphs in §13.4.

In §13.5 we discuss several non-linear notions of 'type p' for metric spaces, notably the notion of Markov type p, and we prove the recent result from [357] that p-uniformly smooth implies Markov type p. The proof uses martingale inequalities for martingales naturally associated to Markov chains on finite state spaces.

In the final chapter, Chapter 14, we briefly outline the recent developments of 'non-commutative' martingale inequalities initiated in [401]. This is the subject of a second volume titled *Martingales in Non-Commutative L_p-Spaces* to follow the present one (to be published on the author's web page). We only give a glimpse of what this is about.

There the probability measure is replaced by a standard normalized trace τ on a von Neumann algebra A, the filtration becomes an increasing sequence (A_n) of von Neumann subalgebras of A, the space $L_p(\tau)$ is now defined as the completion of A equipped with the norm $x \mapsto (\tau(|x|^p))^{1/p}$ and the conditional expectation $\mathbb{E}_n$ with respect to A_n is the orthogonal projection from $L_2(\tau)$ onto the closure of A_n in $L_2(\tau)$. Although there is no analogue of the maximal function (in fact, there are no functions at all !), it turns out that there is a satisfactory non-commutative analogue (by duality) of Doob's inequality (see [292]). Moreover, Gundy's decomposition can be extended to this framework (see [373, 410, 411]). Thus one obtains a version of Burkholder's martingale transform inequalities. In other words, martingale difference sequences in non-commutative L_p-spaces ($1 < p < \infty$) are unconditional. This implies in particular that the latter spaces are UMD as Banach spaces. The Burkholder-Rosenthal and Stein inequalities all have natural generalizations to this setting.