MARTINGALES IN BANACH SPACES

This book is focused on the major applications of martingales to the geometry of Banach spaces, but a substantial discussion of harmonic analysis in Banach space valued Hardy spaces is presented. Exciting links between super-reflexivity and some metric spaces related to computer science are covered, as is an outline of the recently developed theory of non-commutative martingales, which has natural connections with quantum physics and quantum information theory.

Requiring few prerequisites and providing fully detailed proofs for the main results, this self-contained study is accessible to graduate students with basic knowledge of real and complex analysis and functional analysis. Chapters can be read independently, each building from introductory notes, and the diversity of topics included also means this book can serve as the basis for a variety of graduate courses.

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Martingales in Banach Spaces

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Introduction

Martingales (with discrete time) lie at the centre of this book. They are known to have major applications to virtually every corner of probability theory. Our central theme is their applications to the geometry of Banach spaces.

We should emphasize that we do *not* assume any knowledge about scalar valued martingales. Actually, the beginning of this book gives a self-contained introduction to the basic martingale convergence theorems for which the use of the norm of a vector valued random variable instead of the modulus of a scalar one makes little difference. Only when we consider the 'boundedness implies convergence' phenomenon does it start to matter. Indeed, this requires the Banach space B to have the Radon-Nikodým property (RNP). But even at this point, the reader who wishes to concentrate on the scalar case could simply assume that B is finite-dimensional and disregard all the infinite-dimensional technical points. The structure of the proofs remains pertinent if one does so. In fact, it may be good advice for a beginner to do a first reading in this way. One could argue similarly about the property of 'unconditionality of martingale differences' (UMD): although perhaps the presence of a Banach space norm is more disturbing there, our reader could assume at first reading that B is a Hilbert space, thus getting rid of a number of technicalities to which one can return later.

A major feature of the UMD property is its equivalence to the boundedness of the Hilbert transform (HT). Thus we include a substantial excursion in (Banach space valued) harmonic analysis to explain this.

Actually, connections with harmonic analysis abound in this book, as we include a rather detailed exposition of the boundary behaviour of B-valued harmonic (resp. analytic) functions in connections with the RNP (resp. analytic RNP) of the Banach space B. We introduce the corresponding B-valued Hardy spaces in analogy with their probabilistic counterparts. We are partly motivated by the important role they play in operator theory, when one takes for B the space of bounded operators (or the Schatten p-class) on a Hilbert space.

Hardy spaces are closely linked with martingales via Brownian motion: indeed, for any B-valued bounded harmonic (resp. analytic) function u on the unit disc D, the composition $(u(W_{t\wedge T}))_{t>0}$ of u with Brownian motion stopped before it exits D is an example of a continuous B-valued martingale, and its boundary behaviour depends in general on whether B has the RNP (resp. analytic RNP). We describe this connection with Brownian motion in detail, but we refrain from going too far on that road, remaining faithful to our discrete time emphasis. However, we include short sections summarizing just what is necessary to understand the connections with Brownian martingales in the Banach valued context, together with pointers to the relevant literature. In general, the sections that are a bit far off our main goals are marked by an asterisk. For instance, we describe in §7.1 the Banach space valued version of Fefferman's duality theorem between H^1 and BMO. While this is not really part of martingale theory, the interplay with martingales, both historically and heuristically, is so obvious that we felt we had to include it. The asterisked sections could be kept for a second reading.

In addition to the RN and UMD properties, our third main theme is super-reflexivity and its connections with uniform convexity and smoothness. Roughly, we relate the geometric properties of a Banach space B with the study of the p-variation

$$S_p(f) = \left(\sum\nolimits_1^\infty \|f_n - f_{n-1}\|_B^p\right)^{1/p}$$

of B-valued martingales (f_n). Depending on whether $S_p(f) \in L_p$ is necessary or sufficient for the convergence of (f_n) in $L_p(B)$, we can find an equivalent norm on B with modulus of uniform convexity (resp. smoothness) 'at least as good as' the function $t \to t^p$.

We also consider the strong p-variation

$$V_p(f) = \sup_{0=n(0)<n(1)<n(2)<\cdots} \left(\sum\nolimits_1^\infty \|f_{n(k)} - f_{n(k-1)}\|_B^p\right)^{1/p}$$

of a martingale. For that topic (exceptionally) we devote an entire chapter only to the scalar case. Our crucial tool here is the 'real interpolation method'. Real and complex interpolation in general play an important role in L_p-space theory, so we find it natural to devote a significant amount of space to these two 'methods'.

We allow ourselves several excursions aiming to illustrate the efficiency of martingales, for instance to the concentration of measure phenomenon. We also describe some exciting recent work on non-linear properties of metric spaces analogous to uniform convexity/smoothness and type for metric spaces.

We originally intended to include in this book a detailed presentation of 'non-commutative' martingale theory, but that part became so big that we decided to make it the subject of a (hopefully forthcoming) separate volume to be published, perhaps on the author's web page. We merely outline its contents in the last chapter, devoted to non-commutative L_p-spaces. There the complex interpolation method becomes a central tool.

The book should be accessible to graduate students, requiring only the basics of real and complex analysis (mainly Lebesgue integration) and basic functional analysis (mainly duality, the weak and strong topologies and reflexivity of Banach spaces). Our choice is to give fully detailed proofs for the main results and to indicate references to the refinements in the 'Notes and Remarks' or the asterisked sections. We strive to make the presentations self-contained, and when given a choice, we opt for simplicity over maximal generality. For instance, we restrict the Banach space valued harmonic analysis to functions with domains in the unit disc D or the upper half-plane U in $\mathbb{C}$ (or their boundary $\partial D = \mathbb{T}$ or $\partial U = \mathbb{R}$). We feel the main ideas are easier to grasp in the real or complex uni-dimensional case.

The topics (martingales, H^p-space theory, interpolation, Banach space geometry) are quite diverse and should appeal to several distinct audiences. The main novelty is the choice to bring all these topics together in the various parts of this single volume. We should emphasize that the different parts can be read independently, and each time their start is introductory.

There are natural groupings of chapters, such as 1-2-10-11 or 3-4-5-6 (possibly including parts of 1 and 2, but not necessarily), which could form the basis for a graduate course.

Depending on his or her background, a reader is likely to choose to concentrate on different parts. We hope probabilist graduate students will benefit from the detailed introductory presentation of basic H^p-space theory, its connections with martingales, the links with Banach space geometry and the detailed treatment of interpolation theory (which we illustrate by applications to the strong p-variation of martingales), while graduate students with interest in functional analysis and Banach spaces should benefit more from the initial detailed presentation of basic martingale theory. In addition, we hope to attract readers with interest in computer science wishing to see the sources of the various recent developments on finite metric spaces described in Chapter 13. A reader with an advanced knowledge of harmonic analysis and H^p-theory will probably choose to skip the introductory part on that direction, which is written with non-specialists in mind, and concentrate on the issues specific to Banach space valued functions related to the UMD property and the Hilbert transform.

The choice to include so much background on the real and complex interpolation methods in Chapter 8 is motivated by its crucial importance in Banach space valued L_p-space theory, which, in some sense, is the true subject of this book.

Acknowledgement. This book is based on lecture notes for various topics courses given during the last 10 years or so at Texas A&M University. Thanks are due to Robin Campbell, who typed most on them, for her excellent work. I am indebted to Hervé Chapelat, who took notes from my even earlier lectures on H^p-spaces there, for Chapters 3 and 4. The completion of this volume was stimulated by the Winter School on ‘Type, cotype and martingales on Banach spaces and metric spaces’ at IHP (Paris), 2–8 February 2011, for which I would like to thank the organizers. I am very grateful to all those who, at some stage, helped me to correct mistakes and misprints and who suggested improvements of all kinds, in particular Michael Cwikel, Sonia Fourati, Julien Giol, Rostyslav Kravchenko, Bernard Maurey, Adam Osękowski, Javier Parcet, Yanqi Qiu, Mikael de la Salle, Francisco Torres-Ayala, Mateusz Wasilewski, and Quanhua Xu; S. Petermichl for help on Chapter 6; and M. I. Ostrovskii for advice on Chapter 13. I am especially grateful to Mikael de la Salle for drawing all the pictures with TikZ.

Description of the contents

We will now review the contents of this book chapter by chapter.

Chapter 1 begins with preliminary background: we introduce Banach space valued L_p-spaces, conditional expectations and the central notion in this book, namely Banach space valued martingales associated to a filtration $(\mathcal{A}_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We describe the classical examples of filtrations (the dyadic one and the Haar one) in §1.4. If B is an arbitrary Banach space and the martingale (f_n) is associated to some f in $L_p(B)$ by $f_n = \mathbb{E}^{\mathcal{A}_n}(f)$ $(1 \leq p < \infty)$, then, assuming $\mathcal{A} = \mathcal{A}_\infty$ for simplicity, the fundamental convergence theorems say that

$$f_n \to f$$

both in $L_p(B)$ and almost surely (a.s.).

The convergence in $L_p(B)$ is Theorem 1.14, while the a.s. convergence is Theorem 1.30. The latter is based on Doob's classical *maximal inequalities* (Theorem 1.25), which are proved using the crucial notion of *stopping time*. We also describe the dual form of Doob's inequality due to Burkholder-Davis-Gundy (see Theorem 1.26). Doob's maximal inequality shows that the convergence of f_n to f in $L_p(B)$ 'automatically' implies a.s. convergence. This, of course, is special to martingales, but in general it requires $p \geq 1$. However, for martingales that are sums of independent, symmetric random variables (Y_n) (i.e. we have $f_n = \sum_1^n Y_k$), this result holds for $0 < p < 1$ (see Theorem 1.40). It also holds, roughly, for $p = 0$. This is the content of the celebrated Ito-Nisio theorem (see Theorem 1.43), which asserts that even a weak form of convergence of the series $f_n = \sum_1^n Y_k$ implies its a.s. norm convergence.

In §1.8, we prove, again using martingales, a version of Phillips's theorem. The latter is usually stated as saying that, if B is separable, any countably additive measure on the Borel σ-algebra of B is 'Radon', i.e. the measure of a Borel